Mechanical energy budget and mixing efficiency for a radiatively heated ice-covered waterbody

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Ice-covered waterbodies are far from being quiescent systems. In this paper, we investigate ice-covered freshwater basins heated by solar radiation that penetrates across waters with temperatures below or near the temperature of maximum density. In this scenario, solar radiation sets a radiative buoyancy flux, Φ_r , that forces increments of temperature/density in the upper fluid volume, which can become gravitationally unstable and drive convection. The goal of this study is twofold. We first focus on formulating the mechanical energy budget, putting emphasis on the conversion of Φ_r to available potential energy, E_a . We find that E_a results from a competition among Φ_r and the irreversible mixing controlled by the diapycnal and the laminar mixing rates, respectively. Secondly, and based on the above result, we introduce an integral formulation of the mixing efficiency to quantify the rate of mixing over the relevant timescale τ , $\eta_c \equiv \Delta E_{b,\tau} / E_{r,\tau}$, where $\Delta E_{b,\tau}$ and $E_{r,\tau}$ are the change of background potential energy and the time-integrated Φ_r over τ . The above definition is applied to estimate η_c for the first time, finding an approximate value of $\eta_c \approx 0.65$. This result suggests that radiatively heated ice-covered waterbodies might be subject to high mixing rates. Overall, the present work provides a framework to examine energetics and mixing in ice-covered waters.

Key words: Mechanical energy budget, mixing efficiency, solar radiation, under-ice flows

1. Introduction

Ice-covered waterbodies show complex circulation patterns characterised by buoyancy-driven processes during late-Winter/early-Spring season (Farmer 1975; Forrest *et al.* 2013; Rizk *et al.* 2014; Salonen *et al.* 2014; Kirillin *et al.* 2015; Bouffard *et al.* 2016; Yang *et al.* 2017). Over this period, the primary forcing mechanism of such systems is the radiative buoyancy flux, Φ_r , controlled by solar radiation (daylight) able to penetrate across the ice-sheet and heat upper waters. The combination of the radiative heating and the ice-boundary on the top of the waterbody imprints a unique signature in the vertical temperature distribution (figure 1a). The average water temperature profile is characterised by a stable thermal boundary layer (TBL) with temperatures increasing from 0°C at the ice-water interface toward the interior and can extend from tens of

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centimetres to a couple of metres (Mironov *et al.* 2002). The TBL is usually joined to a well-mixed layer, resulting from radiatively-driven convection (RDC), and whose thickness can take the remaining water column or end at a depth where deep stratification might be found, with temperatures below or near the maximum water density, T_{md} (see figure 1a and Farmer 1975; Kirillin *et al.* 2012; Bouffard *et al.* 2018). In this thermal scenario, increases in water temperature lead to rises in water density; the above makes Φ_r an effective driver in ice-covered waters.

Despite the relevance of Φ_r as a fundamental driver in ice-covered waterbodies, there is a lack of understanding of its role in the mechanical energy budget of such systems. Indeed, few studies have accounted for Φ_r in the kinetic energy balance to investigate entrainment driven by RDC in ice-covered waterbodies (Farmer 1975; Mironov *et al.* 2002; Jonas *et al.* 2003). Yet, the mixing efficiency in these systems and its relation with Φ_r are still unknown. Addressing this twofold problem is critical to quantify the energy available to drive motion and enhance mixing. The present work provides a better understanding of the role of Φ_r in the mechanical energy and the mixing of radiatively heated ice-covered freshwater basins. To do so, we formulate a mechanical energy budget and the mixing efficiency for such systems, illustrating its applicability to ice-covered lakes. We consider a hypothetical ice-covered basin containing a Boussinesq fluid that is forced by an unsteady, periodic and volumetric radiative buoyancy flux, Φ_r . The above simplification enables a clear physical interpretation of how Φ_r works in the investigated system.

In §2, we focus on the pathway of Φ_r in the rate of change of mechanical energy E_M ,

$$\dot{E}_M \equiv \dot{E}_k + \dot{E}_p, \quad (1.1)$$

the sum of the kinetic energy, E_k , and the gravitational potential energy, E_p , where $\dot{\xi}$ represents the time derivative of the function ξ . In particular, we examine the role of Φ_r in the rate of change of E_p . To do so, E_p is partitioned as the sum of the background potential energy, E_b , and the available potential energy, E_a (Winters *et al.* 1995). E_b is the minimum potential energy assuming that the entire domain is adiabatically sorted into a stable density distribution, while E_a is the amount of potential energy able to drive motion and mechanical energy exchanges in the system. Hence, the energy controlling the under-ice flow is the ‘available mechanical energy’, $E_{AM} \equiv E_k + E_a$ (Tailleux 2009). In this framework, we explicitly derive $\dot{E}_a$ in terms of Φ_r . The framework reveals that Φ_r is the only energy source that competes against dissipation to sustain E_{AM} in an ice-covered enclosed waterbody.

In §3, we formulate the mixing efficiency, η , based on the most general definition (e.g., Manins & Turner 1978; Gregg *et al.* 2018),

$$\text{Mixing efficiency} \equiv \frac{\text{Change in background potential energy due to mixing}}{\text{Energy input}}. \quad (1.2)$$

Here, it is assumed that the radiative energy working through the water column during the daytime is the only source of energy in the ice-covered basin (figure 1a). We adopt the ‘cumulative’ definition of η (Peltier & Caulfield 2003), η_c , to quantify the net mixing efficiency over a relevant timescale. To ground this formulation, we suggest a methodology for computing η_c based on field observations. In §4, we illustrate the applicability and constraints of this methodology, using observations of the meaningful physical variables in an ice-covered lake. We also provide a first estimation of η_c in a natural system.

We discuss our findings in terms of reported mixing efficiency rates that span from 1/6 in shear-driven stratified flows (e.g., Wüest & Lorke 2003; Peltier & Caulfield 2003), near or higher than 1/2 in buoyancy-driven flows (Wells & Sherman 2001; Hughes *et al.* 2013;

FIGURE 1. (a) Schematic of an ice-covered enclosed waterbasin forced by solar radiation. Green line denotes the vertical density distribution, ρ , of a Boussinesq fluid, so its temperature profile follows a similar shape. Red curve denotes the vertical distribution of radiative induced density flux, F_r , while blue curves schematise flow motions including convective cells, lateral currents and internal wave fields. (b) Characterisation of the energetics involved in the mechanical energy budget (developed in §2) along with the diagram of the energy pathway of the radiatively-forced Boussinesq fluid contained in an ice-covered basin. (1) The radiative buoyancy flux Φ_r increases density in the upper water column driving convection that triggers the (2) buoyancy flux Φ_z , enhancing thus (3) the diapycnal mixing Φ_d and (4) the kinetic energy dissipation rate ϵ . All these processes affect the background stratification and, as a consequence, (5) the laminar mixing rate Φ_i and (6) the ice-water buoyancy flux Φ_{b*} (extended description in §2.2). Red box highlights the cumulative mixing efficiency, η_c , introduced in §3.1.

Davies Wykes & Dalziel 2014; Davies Wykes *et al.* 2015), to extreme mixing regimes associated with the ocean overturning circulation (e.g., Barkan *et al.* 2013; Gayen *et al.* 2014; Vreugdenhil *et al.* 2016; Sohail *et al.* 2018) where predicted rates are close to 1.

2. Mechanical energy budget

We consider an incompressible Boussinesq fluid, with kinematic viscosity ν and molecular mass diffusivity κ contained in an enclosed ice-covered rotating basin of volume V bound by a solid (no-slip) surface $S = A_{ice} + S_b$, where A_{ice} and S_b denote the ice-surface and bottom boundary, respectively (figure 1a). Here, A_{ice} is the only boundary allowing diffusive fluxes across it. The equations of motion can then be expressed as follows

$$\nabla \cdot \mathbf{v} = 0, \quad \dot{\rho} = -\nabla \cdot (\mathbf{v}\rho - \kappa \nabla \rho) + \nabla F_r, \quad (2.1a)$$

$$\dot{\mathbf{v}} = -\nabla (p/\rho_0 + \mathbf{v}^2/2) + (\rho/\rho_0)\mathbf{g} - f\hat{\mathbf{k}} \times \mathbf{v} + \nabla (\nu \nabla \mathbf{v}), \quad (2.1b)$$

where $\mathbf{v} = u\hat{\mathbf{i}} + v\hat{\mathbf{j}} + w\hat{\mathbf{k}}$ is the velocity vector and $z = \mathbf{x} \cdot \hat{\mathbf{k}}$ is the vertical axis positive upward with origin at the ice-water interface. We define the time- and spatial-dependent radiative induced density flux $F_r = F_r(\mathbf{x}, t)$ that works on the fluid. Thus, ∇F_r defines the divergence of the flux F_r that affects the density distribution, ρ , by changes in the temperature field. A general description of the equation of state for Boussinesq fluids and its effect on F_r is provided in the appendix A. The remaining terms in (2.1) have the conventional meanings.

2.1. Energetics of a radiatively-forced ice-covered waterbody

Mass and momentum budgets in (2.1) allow a simple derivation of the mechanical energy budget, $\dot{E}_M$, that involves the rate of change of E_k and E_p . Here, we examine the energy quantities per unit mass, $\rho_0 V$, adopting the standard Φ -notation to denote energetics in $\dot{E}_M$ (Winters *et al.* 1995, among others). To derive the kinetic energy budget,

96 $\dot{E}_k$, we multiply (2.1b) by the velocity field $\mathbf{v}$. Then, using continuity, $\nabla \cdot \mathbf{v} = 0$, and the
 97 boundary conditions on S , the rate of change of $E_k \equiv (1/2)\langle \mathbf{v}^2 \rangle_V$ in V is reduced to

$$\dot{E}_k = - \underbrace{\frac{g}{\rho_0} \langle \rho w \rangle_V}_{\Phi_z} - \underbrace{\nu \langle |\nabla u|^2 + |\nabla v|^2 + |\nabla w|^2 \rangle_V}_{\epsilon}, \quad (2.2)$$

98 where the rate of work across S is null. Φ_z is the volume-averaged, $\langle \cdot \rangle_V$, reversible
 99 buoyancy flux, and ϵ is the volume-averaged kinetic energy dissipation rate. Similarly,
 100 we multiply the mass transport equation (2.1a) by ' gz ' and integrate over V to derive
 101 the rate of change of potential energy $E_p \equiv (g/(\rho_0 V_S)) \langle z \rho \rangle_V$ in V ,

$$\dot{E}_p = \underbrace{\Phi_z + \frac{\kappa g}{\rho_0 V} \oint_S z \nabla \rho \cdot \hat{\mathbf{n}} dS}_{\Phi_b} + \underbrace{- \frac{\kappa g}{\rho_0} \left\langle \frac{d\rho}{dz} \right\rangle_V}_{\Phi_i} + \underbrace{\frac{g}{\rho_0} \left\langle z \nabla F_r \right\rangle_V}_{\Phi_r}, \quad (2.3)$$

102 where $\hat{\mathbf{n}}$ is the outward normal vector to the surface S , Φ_b is the rate of diffusive buoyancy
 103 flux across the surface S , Φ_i is the rate at which E_p increases due to vertical mass
 104 redistribution resulting from molecular diffusion (conversion rate from internal energy,
 105 E_i , to E_p), and Φ_r is the conversion rate from the radiative heating to E_p . Understanding
 106 the role and contribution of Φ_r in $\dot{E}_M$ is the main focus of §2. The potential energy
 107 can be decomposed as a background and an available component, $E_p = E_b + E_a$. The
 108 background potential energy, $E_b \equiv (g/(\rho_0 V_S)) \langle z_* \rho \rangle_V$, is defined in terms of the global
 109 reference position $z_*(\mathbf{x}, t)$ (Winters *et al.* 1995) that characterises the height to which
 110 each fluid parcel is allocated to reach its stable position in the volume V . The evolution
 111 equation of E_b in V is given by

$$\dot{E}_b = \underbrace{\frac{\kappa g}{\rho_0 V_S} \oint_S z_* \nabla \rho \cdot \hat{\mathbf{n}} dS}_{\Phi_{b_*}} + \underbrace{- \frac{\kappa g}{\rho_0} \left\langle \frac{dz_*}{d\rho} |\nabla \rho|^2 \right\rangle_V}_{\Phi_d}, \quad (2.4)$$

112 where Φ_{b_*} is the rate of change of E_b due to diffusive buoyancy fluxes across A_{ice} , and
 113 Φ_d is the rate of change of E_b resulting from irreversible mixing of the density field in V .
 114 The evolution equation for E_a results from subtracting (2.4) from (2.3),

$$\dot{E}_a = \Phi_r + \Phi_z + (\Phi_b - \Phi_{b_*}) - (\Phi_d - \Phi_i). \quad (2.5)$$

115 We stress that the surface buoyancy flux Φ_b is defined null across S_b but might differ
 116 from zero across the ice-surface boundary A_{ice} , at $z = z_{\text{ice}}$. Thus

$$\Phi_b = \frac{\kappa g}{\rho_0 V} \oint_{A_{\text{ice}}} z \frac{\partial \rho}{\partial z} dS = \frac{\kappa g A_{\text{ice}} z_{\text{ice}}}{\rho_0 V} \left\langle \frac{\partial \rho}{\partial z} \right\rangle_{A_{\text{ice}}}. \quad (2.6)$$

117 Similarly, the surface buoyancy flux associated with the evolution of E_b at the level z_* is
 118 determined by

$$\Phi_{b_*} = \frac{\kappa g}{\rho_0 V} \oint_{A_{\text{ice}}} z_* \frac{\partial \rho}{\partial z} dS = \frac{\kappa g A_{\text{ice}} z_*(z_{\text{ice}})}{\rho_0 V} \left\langle \frac{\partial \rho}{\partial z} \right\rangle_{A_{\text{ice}}}. \quad (2.7)$$

119 Thus, we can express the net surface buoyancy flux, $(\Phi_b - \Phi_{b_*})$, across the ice-boundary
 120 working on the rate of change of E_a , as

$$\Phi_b - \Phi_{b_*} = [z_{\text{ice}} - z_*(z_{\text{ice}})] \frac{\kappa g A_{\text{ice}}}{\rho_0 V} \left\langle \frac{\partial \rho}{\partial z} \right\rangle_{A_{\text{ice}}}. \quad (2.8)$$

However, since fluid parcels in the vicinity of the ice-water interface become lighter and strongly stable as $z \rightarrow z_{\text{ice}}$, the reference height of a fluid parcel of density $\rho(z \approx z_{\text{ice}})$ is $z_*(z \approx z_{\text{ice}}) = z_{\text{ice}}$. This fundamental thermodynamic characteristic that the ice-water boundary imposes to the fluid stability leads to the following equality:

$$\Phi_{b_*}|_{A_{\text{ice}}} = \Phi_b|_{A_{\text{ice}}}. \quad (2.9)$$

Hence, by (2.9), the net surface buoyancy flux (2.8) is null and therefore plays no role in $\dot{E}_a$ (2.5). Notice that $(\partial\rho/\partial z)|_{z=z_{\text{ice}}} \leq 0$, so $\Phi_{b_*}|_{z=z_{\text{ice}}} = \Phi_b|_{z=z_{\text{ice}}} \leq 0$ is a negative flux that decreases both E_p and E_b equally.

2.2. The pathway of Φ_r in the mechanical energy budget

The evolution equation for the mechanical energy results by merging (2.2) and (2.3),

$$\dot{E}_M = \dot{E}_k + \dot{E}_p = -\epsilon + \Phi_b + \Phi_i + \Phi_r. \quad (2.10)$$

Replacing $\Phi_b = \Phi_{b_*} = \dot{E}_b - \Phi_d$ into (2.10), we obtain the evolution equation for the available mechanical energy, E_{AM} ,

$$\dot{E}_{AM} \equiv \dot{E}_k + \dot{E}_a = \dot{E}_M - \dot{E}_b = -\epsilon - (\Phi_d - \Phi_i) + \Phi_r. \quad (2.11)$$

The energy budget in (2.11) retains the elemental energetics governing the loss and gain of available mechanical energy in the system. From (2.11), we distinguish that ϵ represents the net decay rate of E_k , while $(\Phi_d - \Phi_i) - \Phi_r$ represents the net rate of change (either decay or gain) of E_a in the system. Thus, the evolution of E_a results from the competition between its decay rate, controlled by the action of irreversible mixing, $\Phi_d - \Phi_i$, and its increment, resulting from the conversion of radiative to potential energy, Φ_r . If Φ_r becomes null, then E_{AM} will decay in time. Therefore, it is inferred that Φ_r is the fundamental generator of E_a and the implicit driver for E_k .

The pathway of radiative energy flux, Φ_r , into the mechanical energetics in an ice-covered enclosed waterbody is illustrated in figure 1b. The energy pathway is described by six main processes. (1) During the daytime, Φ_r works over upper waters driving localised increments of temperature and density leading to unstable density distributions and a gain of E_a . (2) Part of E_a is transformed to buoyancy flux Φ_z via mechanical convection that tries to restore the system to a stable density distribution. In this process, Φ_z is transformed into E_k , energising the vertical and horizontal transport. (3) Simultaneously, diapycnal mixing Φ_d transfers E_a toward E_b , and this process is enhanced once convection starts stirring the well-mixed layer and deep stratified regions. (4) Convection also enhances the rate of decay of kinetic energy, ϵ , by friction which is transformed into internal energy. (5) E_p increases at the expense of the internal energy contained in the stratified fluid at a rate of Φ_i . (6) Finally, the system loses E_b via the surface buoyancy flux at a rate of Φ_{b_*} controlled by the density gradient in the TBL. Thus, the energy pathway in our system can be understood as a spiral cycle characterised by the gain of E_{AM} (which, as a consequence, enhances the rate of increment of E_b) as long as the radiative buoyancy flux, Φ_r , works on the fluid. At night, $\Phi_r = 0$ and E_{AM} starts decaying until the beginning of the next daytime.

We do not examine the internal energy reservoir nor its link with $\dot{E}_M$. For an extensive analysis of the internal energy decomposition, we refer to [Tailleux \(2009\)](#). Additionally, although we do not develop a radiative energy balance, it is assumed that Φ_r results from the net component of the radiative energy that reaches the volume V . The radiative balance (including downward/upward fluxes) on ice-covered lakes is addressed by [Kirillin et al. \(2012\)](#) and [Bouffard et al. \(2016\)](#).

3. Mixing efficiency

We now turn our focus to characterising the mixing efficiency in the system described in figure 1, based on (1.2). We might be tempted to adopt the classic definition of instantaneous mixing efficiency (Peltier & Caulfield 2003) in our system,

$$\eta \equiv \frac{\Phi_d - \Phi_i}{(\Phi_d - \Phi_i) + \epsilon}. \quad (3.1)$$

However, in the system analysed here, the only source of external energy is the radiative energy, E_r , that works on the volume via the radiative buoyancy flux, Φ_r . Hence, during the daytime, the instantaneous mixing efficiency is defined as

$$\eta \equiv \frac{\Phi_d - \Phi_i}{\Phi_r}, \quad (3.2)$$

which compares the rate of irreversible mixing to the rate of radiative energy injected into the volume V subject to mixing. Yet, the definition (3.2) is only valid when $\Phi_r \neq 0$ (daytime phase), and not when $\Phi_r = 0$ (night-time phase).

Mixing efficiency can be further examined in a specific energy regime. If the rate of radiative energy supplied to the system is balanced by the sum of the rate of decay of kinetic energy and the rate of mixing, the available mechanical energy budget in (2.11) can reach a steady state regime that leads to

$$\dot{E}_{AM} = -\epsilon - (\Phi_d - \Phi_i) + \Phi_r = 0. \quad (3.3)$$

In the steady energy regime (3.3), expressions (3.1) and (3.2) become equivalent since $\Phi_r = (\Phi_d - \Phi_i) + \epsilon$. Dividing equation (3.3) by Φ_r , yields

$$\eta = 1 - \frac{\epsilon}{\Phi_r}. \quad (3.4)$$

However, given the unsteady but periodic nature of Φ_r , the equilibrium state in (3.3) might be unrealistic in ice-covered lakes. Therefore, the mixing efficiency requires a more robust formulation that is able to incorporate the effect of time-dependent radiative forcing over the relevant timescale. To achieve a more robust formulation of the mixing efficiency, a time-integrated definition is used.

3.1. Cumulative mixing efficiency

We define the ‘cumulative mixing efficiency’ (e.g., Manins & Turner 1978; Peltier & Caulfield 2003; Davies Wykes & Dalziel 2014) as

$$\eta_c \equiv \frac{\int_t^{t+\tau} \dot{E}_b dt'}{\int_t^{t+\tau} \Phi_r(t') dt'} \equiv \frac{\Delta E_{b,\tau}}{E_{r,\tau}}, \quad (3.5)$$

where $\Delta E_{b,\tau} = E_b(t + \tau) - E_b(t)$ corresponds to the change of E_b over a timescale τ , while $E_{r,\tau}$ is the radiative energy injected into the bulk via radiative buoyancy flux, Φ_r , during τ . Notice that (3.5) will not be equal to the average of (3.4) over the timescale τ unless the system is in steady state.

The ratio η_c is straightforward to compute if the whole density field and radiative flux are known. Numerical experiments provide a robust approach for examining both η and η_c in ice-covered flows forced by Φ_r . However, considering the standard methodologies used to obtain in-situ observations, the ‘cumulative’ formulation fits better to estimate the rate of mixing in natural systems.

3.2. Cumulative mixing efficiency based on vertically-integrated observations

We formulate a simple, handy methodology to estimate η_c based on in-situ observations of density ρ (obtained from casts of temperature, T , and conductivity, C , as a function of depth), solar irradiance at the ice-water interface, H_0 , and light distribution across the water column in a basin of volume V . For simplicity, we consider that H_0 is uniformly distributed over the ice-water interface of area A_{ice} and that the hypsometric curve is known, $\mathcal{A}(z)$, which characterises the horizontal area of the waterbody as a function of depth z . Notice that $\mathcal{A}(z = z_{\text{ice}}) = A_{\text{ice}}$.

Let us assume we measure N_p vertical density profiles, ρ^j ($j = 1, \dots, N_p$), throughout V . Furthermore, consider that each profile spans from z_{ice} to $z_{\text{bot}}^{(j)}$, where $z_{\text{bot}}^{(j)}$ might be different for each profile j . The cumulative mixing efficiency can be estimated using the following steps:

- (i) Sort all the density profiles ρ^j to obtain a unique stable density distribution ρ_* increasing monotonically with depth from the top z_{ice} to the deepest bottom D .
- (ii) Interpolate ρ_* into an equally spaced reference state z_* that allocates fluid parcels at a stable position. This procedure establishes a one-to-one function between z_* and ρ_* .
- (iii) Estimate E_b (J kg^{-1}) based on $\rho_*(z_*, t)$ and the hypsometric curve $\mathcal{A}(z_*)$,

$$\langle E_b \rangle_{N_p} \approx \frac{g}{\rho_0 V} \int_D^{z_{\text{ice}}} z_*(t) \rho_*(z_*, t) \mathcal{A}(z_*) dz_*. \quad (3.6)$$

- (iv) Estimate the solar irradiance flux at the ice-water interface, $H_0(t)$ (W m^{-2}), and the light distribution over the water column. From the light distribution, estimate a bulk e-folding decay depth-scale, λ . Then, the radiative induced density flux, $F_r(z, t)$ ($\text{kg m}^{-2} \text{s}^{-1}$), can be modelled as

$$F_r(z, t) \approx F_0(z, t) e^{z/\lambda}, \quad (3.7)$$

where $F_0 = (\alpha_T/c_p) H_0(t)$, while α_T and c_p are the thermal expansion coefficient and the heat capacity of water, respectively. Note that the procedure is not restricted to an exponential decrease of light but rather applies to any vertical light absorption distribution.

- (v) Estimate the radiative buoyancy flux (W kg^{-1}):

$$\Phi_r(t) \approx \frac{g}{\rho_0} \frac{1}{D} \int_D^{z_{\text{ice}}} z \frac{dF_r}{dz} dz. \quad (3.8)$$

- (vi) Estimate the time-integrated radiative energy over a period τ , $E_{r,\tau}$ (J kg^{-1}) defined in (3.5).

- (vii) The cumulative mixing efficiency over a period τ is estimated using (iii) and (vi):

$$\eta_c \approx \frac{\langle E_b(t + \tau) \rangle_{N_p} - \langle E_b(t) \rangle_{N_p}}{E_{r,\tau}}. \quad (3.9)$$

It must be stressed that η_c relies on a volumetric integration. In effect, η_c will depend on the radiative flux strength and its vertical distribution, the background stratification (density/temperature structure), but also on the basin topography. Basin topography can boost the formation of cross-shore buoyancy gradients and consequently induce horizontal flows. Still, if there is little horizontal variability in the density field and the density at the lowest measured point used to estimate $\langle E_b \rangle_{N_p}$ is larger than the highest density in the unstable layer near the ice interface, then the introduced field-based method and the theoretical formulation in §3.1 would be equivalent. In contrast, under-sampling in a system with strong horizontal variability can lead to sub- or overestimation of the global

FIGURE 2. Observations in the ice-covered Lake Onego (Russia) from 12th to 20st of March 2017. (a) Solar irradiance H_0 in the water column, (b) density field, $\rho(z, t)$, relative to the density at the ice-water interface, ρ_0 . (c) Background potential energy $\langle E_b \rangle_{N_p}$ (blue line, left vertical axis), linear fitting (black dashed-line, left vertical axis), diurnally-averaged $\langle E_b \rangle_{N_p}$ (green circles, left vertical axis) and radiative energy, $E_{r,\tau}$ (red squares, right vertical axis).

mixing efficiency rate (Davies Wykes *et al.* 2015). Thus, the better the spatial resolution of the observations, the more robust the estimation of η_c is, assuming that the system is closed and/or neglects other forcing mechanisms.

This paper does not aim to give practical information about techniques required for surveying under ice-covered waterbodies. A detailed analysis of under-ice measurements and the errors resulting from sampling is presented by Bouffard *et al.* (2018).

4. Application: observations in an ice-covered lake

The methodology developed in §3.2 is applied to estimate η_c using recent in-situ, high-resolution observations from Lake Onego (Russia) during late-Winter 2017 (Bouffard *et al.* 2018). Here we consider the simplest scenario of one monitoring spot. Observations of under-ice solar irradiance, light, and density distribution over the water column allowed estimating the evolution of radiative buoyancy flux, Φ_r , using (3.8) and the background potential energy, E_b , using (3.6). During this time, light penetrated between 1 to 1.5 m deep and the solar irradiance, H_0 , at the ice-water interface reached up to 100 W m⁻² (figure 2a). The vertical density distribution (figure 2b) is well-characterised by vertical

signature schematised in figure 1. It has a stable upper thermally-controlled boundary layer (< 1 m) joined to a well-mixed layer attributed to radiatively-driven convection (~ 10 m), which in turn is lower-bound by a weakly stratified layer. Temperatures spanned from $0-1.5$ °C (from top to bottom), and conductivity showed low and constant magnitudes (27.00 ± 0.07 $\mu\text{S cm}^{-1}$, ≈ 0.02 PSU) over the measuring period.

Figure 2c shows the temporal evolution of the estimated background potential energy $\langle E_b \rangle_{N_p}$ (blue line linked to left vertical axis in units $+101$ J kg^{-1}). It is characterised by an average increment of 8×10^{-5} $\text{J kg}^{-1} \text{ day}^{-1}$ (dashed-line) and diurnally-averaged values, $\langle E_b \rangle_{N_p, \tau}$, in green circles. Also, red squares denote the diurnally-integrated radiative buoyancy flux, $E_{r, \tau}$ (right vertical axis in units $\times 10^{-4}$ J kg^{-1}). The cumulative mixing efficiency over the observed period, $\tau = 8$ days, was estimated as in (3.9), obtaining a value of $\eta_c \approx 0.65$. We also estimated η_c at diurnal scale, $\tau = 1$ day, finding a median of 0.74 and a standard deviation of 0.19. Despite this difference, we emphasise that both values of η_c are higher than the mixing efficiency broadly reported in stratified shear flows (Peltier & Caulfield 2003; Wüest & Lorke 2003; Gregg *et al.* 2018), while the eight-days-based value of η_c (as well as the its median value) is remarkably close to those values found in buoyancy-driven mixing (e.g., Hughes *et al.* 2013; Davies Wykes & Dalziel 2014; Sohail *et al.* 2018). Indeed, our results are consistent with the theoretical findings of Davies Wykes *et al.* (2015) who showed that in buoyancy-driven mixing η_c can be higher than $1/2$ if there is stratification below the mixed layer. Longer time series and denser monitoring in smaller ice-covered lakes may be able to provide statistically more accurate estimations of η_c in the future.

5. Concluding remarks

This paper introduces a framework to examine the mechanical energy budget and the mixing efficiency of radiatively-forced, ice-covered Boussinesq flows. The theoretical formulation relies on the following assumptions:

- The fluid is confined to an ice-covered enclosed basin, meaning there is no advective exchange across boundaries. The model can be expanded to include (advective/diffusive) boundary fluxes in mechanical energy components. For instance, the bottom boundary could be forced by geothermal fluxes (Kirillin *et al.* 2012; Rizk *et al.* 2014).
- The Boussinesq density increases with temperature so $\rho \leq \rho_{\max}$ (freshwater basin). However, the presence of fully-mixed water constituents (e.g., salinity) does not change the theoretical framework. Different equations of state (see appendix A) can be investigated via numerical approaches (ongoing work).
- Radiative-forcing is parametrised by a monochromatic light attenuation depth-scale, λ . More sophisticated models considering the light-spectrum distribution can be readily adopted (Mironov *et al.* 2002; Bouffard *et al.* 2018) and would change the z -dependent function in F_r (3.7).

This work has three key outcomes:

- In §2, we show that the radiative buoyancy flux, Φ_r , is a source of available mechanical energy and the fundamental driver in an ice-covered enclosed waterbody.
- In §3, we formulate the cumulative mixing efficiency η_c to quantify the change in background potential energy from the supplied radiative energy at relevant timescales. We further suggest a handy field-based methodology to estimate η_c providing its scope.
- Results in §4 illustrate a practical application of our framework and present a first estimation of the cumulative mixing efficiency in an ice-covered lake, $\eta_c \approx 0.65$. This value indicates that RDC in ice-covered lakes might be attributed to strong mixing regimes.

Still, similar to shear-driven mixing, depending on the integration timescale, τ , and the turbulence age (Mashayek *et al.* 2017), η_c might vary over the course of the ice-covered season. The potential variability of η_c is fostered by the fact that the background temperature/density distributions is changing in time. This change is particularly critical in the well-mixed layer whose temperature tends to $T_{md} \approx 4^\circ\text{C}$. In this scenario, we might expect a drop of η_c because of two reasons. First, the thermal expansion coefficient, $\alpha_T \equiv \rho_0^{-1} \partial \rho / \partial T$ (see appendix A), decays as $T \rightarrow T_{md}$, implying that the radiative buoyancy induced flux, F_r , becomes weaker in magnitude as $T \rightarrow T_{md}$. Secondly, the water column below the TBL tends to become fully mixed (e.g., Kirillin *et al.* 2012; Salonen *et al.* 2014). Then, using the same theoretical arguments by Davies Wykes *et al.* (2015), η_c would tend to 1/2. Certainly, further studies are required for a deeper exploration of mixing regimes in ice-covered systems.

Despite the similarities between the RDC and the classical Rayleigh-Bénard convection (Hughes *et al.* 2013) and Rayleigh-Taylor instability (Davies Wykes & Dalziel 2014), the magnitude of η_c , the forcing mechanism (figure 1b and 2a), and the signature of density profiles shown in figure 2b, suggest that RDC might be controlled by Rayleigh-Taylor like instabilities, especially in zones far from the basin lateral boundaries. In real systems, however, the basin topography can set cross-shore buoyancy gradients and so drive downslope flow (Kirillin *et al.* 2015), similarly to those induced by nighttime cooling in ice-free systems (Wells & Sherman 2001). Moreover, in large lakes, Coriolis force can affect cross-shore flows establishing, for instance, geostrophic circulations and localised gyres (Forrest *et al.* 2013; Rizk *et al.* 2014). In the aforementioned scenario, η_c might show similar values to those reported in studies associated with the ocean overturning circulation (Barkan *et al.* 2013; Vreugdenhil *et al.* 2016; Sohail *et al.* 2018). The energy distribution and the global η_c require to be systematically examined in the various mentioned scenarios. As suggested by Gregg *et al.* (2018) regarding the ocean's mixing problem, future works should further consider parameterisations based on relevant turbulence intensity parameters such as the Rayleigh number, the Reynolds buoyancy number, and the gradient Richardson number.

Quantifying the energetics and mixing in ice-covered waters is not only relevant for inferring the distribution of water constituents, but also for establishing the evolution of the ice-cover (Kirillin *et al.* 2012). Yang *et al.* (2017) and MacKay *et al.* (2017) suggested that strong vertical mixing can enhance the surface buoyancy flux, Φ_b , across the ice-water interface and thus potentially influence the ice-break. In this work, we show that Φ_b relies on the TBL structure under the ice, which in turn depends on the radiative buoyancy induced flux, F_r , and the temperature of the adjacent well-mixed layer (see figure 1b). In particular, we notice that the optical properties of waters can significantly affect the light distribution and hence F_r . In effect, for a same solar irradiance flux reaching the ice-water interface, H_0 , dark colour waters (such as Boreal lakes) tend to concentrate more heat in the very top of the water column than clear waters (such as pristine Alpine lakes). The above fact implies that not only the radiative buoyancy flux, Φ_r , but also Φ_b , can be modified depending on the water colour and the light distribution (Bouffard *et al.* 2018). Investigating the impact of the optical properties of water on Φ_b and Φ_r , and overall on mixing, might provide valuable information for determining the lifespan and sensibility of the ice-cover in waterbodies.

Finally, we argue that the isolated conditions of ice-covered lakes make these systems particularly suitable for connecting lab and numerical studies to field scale. This work provides a framework to continue the exploration of physical processes in ice-covered waterbodies.

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Appendix A. Equation of state and its impact on F_r and Φ_r

The radiative induced density flux, F_r , that forces the mass transport equation (2.1a), depends on the radiative heating distribution across the water column and the thermal expansion coefficient, α_T . The last parameter sets the equation of state (EoS) of the fluid. In the case of a Boussinesq fluid (here, freshwater), a nonlinear EoS can be parametrised as follow:

$$\rho \equiv \rho_0 \mathcal{P}_n(T, \mathbf{x}, t) \quad \text{and} \quad \alpha_T \equiv \rho_0^{-1} \partial \rho / \partial T = \partial \mathcal{P}_n / \partial T = \mathcal{G}_{n-1}(T, \mathbf{x}, t), \quad (\text{A } 1)$$

where $\mathcal{P}_n$ is a dimensionless polynomial of order n , T is the temperature field, α_T is the general thermal expansion coefficient characterised by a temperature-dependent polynomial of order $n - 1$, $\mathcal{G}_{n-1}$. Then, F_r can be expressed as

$$F_r = \alpha_T(T, \mathbf{x}, t) \frac{H_0(t)}{c_p} e^{z/\lambda} = F_0(T, \mathbf{x}, t) e^{z/\lambda}, \quad (\text{A } 2)$$

where $H_0(t)e^{z/\lambda}$ is the temporal and vertical radiative heating distribution and c_p is the heat capacity of water. Thus, the radiative buoyancy flux per unit mass is given by $\Phi_r \equiv g\rho_0^{-1} \langle z \nabla F_r \rangle_V$, where ∇ is the divergence.

In the case of waters found between $0^\circ\text{C} \leq T < 2^\circ\text{C}$, a linear EoS captures reasonably well the relation between temperature and density. In this scenario, α_T is a positive constant. Therefore, F_r can be expressed as $F_r(z, t) = (\alpha_T/c_p) H_0(t)e^{z/\lambda}$, and the radiative buoyancy flux is reduced to $\Phi_r \equiv g\rho_0^{-1} \langle (\alpha_T/c_p) (H_0(t)/\lambda) z e^{z/\lambda} \rangle_V$. However, a quadratic EoS can describe fairly well the range of temperature $0^\circ\text{C} \leq T \leq 4^\circ\text{C}$. In this case, α_T is characterised by a linear function of the temperature, and reaches a minimum at the temperature of maximum density.

REFERENCES

- BARKAN, R., WINTERS, K. & LLEWELLYN SMITH, S. 2013 Rotating horizontal convection. *J. Fluid Mech.* **723**, 556–586.
- BOUFFARD, D., ZDOROVENNOV, R. E., ZDOROVENNOVA, G. E., PASCHE, N., WÜEST, A. & TERZHEVIK, A. Y. 2016 Ice-covered lake Onega: effects of radiation on convection and internal waves. *Hydrobiologia* **780** (1), 21–36.
- BOUFFARD, D., ZDOROVENNOVA, G., BOGDANOV, S., EFREMOVA, T., LAVANCHY, S., PALSHIN, N., TERZHEVIK, A., RÅMAN VINNÅ, L., VOLKOV, S., WÜEST, A., ZDOROVENNOV, R. & ULLOA, H. N. 2018 How to quantify under-ice convection dynamics in lakes. Implication for life under the ice. *Inland Waters* (Accepted under revision).
- DAVIES WYKES, M. & DALZIEL, S. 2014 Efficient mixing in stratified flows: experimental study of a Rayleigh–Taylor unstable interface within an otherwise stable stratification. *J. Fluid Mech.* **756**, 1027–1057.
- DAVIES WYKES, M., HUGHES, G. & DALZIEL, S. 2015 On the meaning of mixing efficiency for buoyancy-driven mixing in stratified turbulent flows. *J. Fluid Mech.* **781**, 261–275.
- FARMER, D. M. 1975 Penetrative convection in the absence of mean shear. *Quart. J. Roy. Meteor. Soc.* **101** (430), 869–891.

- FORREST, A. L., LAVAL, B. E., PIETERS, R. & LIM, D. S. S. 2013 A cyclonic gyre in an ice-covered lake. *Limnol. Oceanogr.* **58** (1), 363–375.
- GAYEN, B., GRIFFITHS, R. & HUGHES, G. 2014 Stability transitions and turbulence in horizontal convection. *J. Fluid Mech.* **751**, 698–724.
- GREGG, M.C., D’ASARO, E.A., RILEY, J.J. & KUNZE, E. 2018 Mixing efficiency in the ocean. *Annu. Rev. Mar. Sci.* **10** (1), 443–473.
- HUGHES, G., GAYEN, B. & GRIFFITHS, R. 2013 Available potential energy in Rayleigh–Bénard convection. *J. Fluid Mech.* **729**.
- JONAS, T., TERZHEVIK, A. Y., MIRONOV, D. V. & WÜEST, A. 2003 Radiatively driven convection in an ice-covered lake investigated by using temperature microstructure technique. *J. Geophys. Res.* **108**, 3183.
- KIRILLIN, G., LEPPÄRANTA, M., TERZHEVIK, A., GRANIN, N., BERNHARDT, J., ENGELHARDT, C., EFREMOVA, T., GOLOSOV, S., PALSHIN, N., SHERSTYANKIN, P., ZDOROVENNOVA, G. & ZDOROVENNOV, R. 2012 Physics of seasonally ice-covered lakes: a review. *Aquat. Sci.* **74** (4), 659–682.
- KIRILLIN, G. B., FORREST, A. L., GRAVES, K. E., FISCHER, A., ENGELHARDT, C. & LAVAL, B. E. 2015 Axisymmetric circulation driven by marginal heating in ice-covered lakes. *Geophys. Res. Lett.* **42** (8), 2893–2900.
- MACKEY, M. D., VERSEGHY, D. L., FORTIN, V. & RENNIE, M. D. 2017 Wintertime simulations of a boreal lake with the canadian small lake model. *J. Hydrometeorol.* **18** (8), 2143–2160.
- MANINS, P. C. & TURNER, J. S. 1978 The relation between the flux ratio and energy ratio in convectively mixed layers. *Quart. J. Roy. Meteor. Soc.* **104** (439), 39–44.
- MASHAYEK, A., SALEHIPOUR, H., BOUFFARD, D., CAULFIELD, C. P., FERRARI, R., NIKURASHIN, M., PELTIER, W. R. & SMYTH, W. D. 2017 Efficiency of turbulent mixing in the abyssal ocean circulation. *Geophys. Res. Lett.* **44** (12), 6296–6306.
- MIRONOV, D., TERZHEVIK, A., KIRILLIN, G., JONAS, T., MALM, J. & FARMER, D. 2002 Radiatively driven convection in ice-covered lakes: Observations, scaling, and a mixed layer model. *J. Geophys. Res.* **107** (C4), 3032.
- PELTIER, W. R. & CAULFIELD, C. P. 2003 Mixing efficiency in stratified shear flows. *Annu. Rev. Fluid Mech.* **35** (1), 135–167.
- RIZK, W., KIRILLIN, G. & LEPPÄRANTA, M. 2014 Basin-scale circulation and heat fluxes in ice-covered lakes. *Limnol. Oceanogr.* **59** (2), 445–464.
- SALONEN, K., PULKKANEN, M., SALMI, P. & GRIFFITHS, R. W. 2014 Interannual variability of circulation under spring ice in a boreal lake. *Limnol. Oceanogr.* **59** (6), 2121–2132.
- SOHAIL, T., GAYEN, B. & HOGG, A. M. 2018 Convection enhances mixing in the southern ocean. *Geophys. Res. Lett.* **45** (9), 4198–4207.
- TAILLEUX, R. 2009 On the energetics of stratified turbulent mixing, irreversible thermodynamics, Boussinesq models and the ocean heat engine controversy. *J. Fluid Mech.* **638**, 339–382.
- VREUGDENHIL, C. A., GAYEN, B. & GRIFFITHS, R. W. 2016 Mixing and dissipation in a geostrophic buoyancy-driven circulation. *J. Geophys. Res. Oceans* **121** (8), 6076–6091.
- WELLS, M. G. & SHERMAN, B. 2001 Stratification produced by surface cooling in lakes with significant shallow regions. *Limnol. Oceanogr.* **46** (7), 1747–1759.
- WINTERS, K., LOMBARD, P., RILEY, J. & D’ASARO, E. 1995 Available potential energy and mixing in density-stratified fluids. *J. Fluid Mech.* **289**, 115–128.
- WÜEST, A. & LORKE, A. 2003 Small-scale hydrodynamics in lakes. *Annu. Rev. Fluid Mech.* **35** (1), 373–412.
- YANG, B., YOUNG, J., BROWN, L. & WELLS, M. 2017 High-frequency observations of temperature and dissolved oxygen reveal under-ice convection in a large lake. *Geophys. Res. Lett.* **44** (24), 12,218–12,226.