Bubble Plume Modeling for Lake Restoration

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A steady bubble plume model is developed to describe a weak air (or oxygen) bubble injection system used for the restoration of deep stratified lakes. Since the model is designed for two modes of operation, i.e., oxygenation and artificial mixing, gas exchange between water and bubbles has to be included. The integral model is based on the entrainment hypothesis and a variable buoyancy flux determined by the local plume properties and the ambient water column. Fluxes of eight properties are described by nonlinear differential equations which can be numerically integrated. In addition, five equations of state are used. The model leaves open two initial conditions, plume radius and plume velocity. Model calculations with real lake water profiles demonstrate the range of applicability for both modes of operation. The model agrees reasonably well with field data and with laboratory experiments conducted by various investigators.

1. INTRODUCTION

In spite of enormous efforts made to fight lake eutrophication, in many lakes concentrations of phosphorus and other planktonic nutrients are still far above critical values [Rast and Lee, 1983]. As a consequence of high primary productivity values, oxygen concentrations in the hypolimnion of eutrophic lakes drop to low values or even to zero. So-called internal lake restoration measures have been designed to improve the hypolimnetic oxygen levels and to limit the recycling of phosphorus from the sediments into the lake water [Imboden, 1985]. Two restoration techniques which can be used separately or in combination are (1) artificial mixing of the water column during the cold season and (2) input of oxygen into the hypolimnion during summer in such a way as to preserve the stratification.

Artificial mixing is technically simpler and also cheaper and is thus usually the first method to be evaluated. It is designed to add extra mixing energy to the lake at the time when the density stratification is weakest in order to allow water exchange down to the very bottom, or, if mixing occurs naturally, to prolong the period of mixing. Complete mixing maximizes the uptake of oxygen by natural gas exchange at the water surface. In fact, incomplete mixing may not only result from low wind exposure but may also be a consequence of lake eutrophication: Mineralization of biomass in the hypolimnion often leads to the steady accumulation of dissolved substances in the deep waters and thus to a chemically induced permanent stratification [Jøller, 1985]. In some lakes, the hypolimnetic oxygen reserves may not be large enough to prevent the hypolimnion from becoming

anoxic during the summer, even if complete saturation had been achieved during the winter. In such situations, oxygen can be artificially introduced into the hypolimnion during the summer. This is commonly combined with artificial mixing during the winter.

Artificial mixing is achieved, in most cases, by injecting compressed air into the deepest part of the lake. However, different technical systems have been designed for the injection of oxygen into a stratified lake. One frequently used system is the Atlas-Copco "Limno" device [Bucksteeg and Hollfelder, 1978]. This system consists of two concentric tubes placed vertically in the water column. Air is pumped into the lower end of the inner tube, in which the water rises, then sinks back through the outer tube and leaves the system through outlets placed at the appropriate depth.

In this article a different approach is described. The system "Tanytarsus," designed by the two Swiss engineers, E. Jungo and U. Schaffner, is presently in operation in several deep Swiss lakes [Imboden, 1985; Stadelmann, 1988; Stöckli and Schmid, 1987]. The system uses the same lake installation for both the injection of compressed air for artificial mixing in winter and the injection of pure oxygen into the hypolimnion in summer (Figures 1a and 1b). Pure oxygen is used for hypolimnetic oxygenation to prevent the accumulation of molecular nitrogen which can be toxic to fish [Fast and Hulquist, 1982].

Both modes of operation are physically similar. The main difference between them consists in the choice of the initial bubble radius and the selection of either natural air or pure oxygen. During the winter the air bubbles have to be large enough to guarantee that the buoyancy force acts up to the lake surface. In contrast, for efficient oxygenation, the oxygen bubbles have to be small enough to dissolve completely in the oxygen-poor hypolimnion before entering the

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Paper number 92WR01681.
0043-1397/92/92WR-01681\$05.00

Fig. 1a. Schematic diagrams of the application of the bubble injection system (below) and of the coupling of the plume model with a water quality lake model (above). The two models should run independently of one another, alternating over a short time interval (steady input) to supply one another with their respective required input variables. Labeled quantities are those appearing in (24) and (25).

oxygen-rich surface waters. In addition, the bubble plume should be weak enough so that the induced water circulation is restricted to the hypolimnion. Otherwise, the induced flux of nutrients from the hypolimnion into the epilimnion could further stimulate primary production in the lake.

Diffusers for both modes of operation have been successfully designed and employed in several medium-sized Swiss lakes (Baldeggersee, Sempachersee, Hallwilersee). In order to optimize design and operation of the installations with respect to oxygen input and energy consumption it is important to gain a better understanding of the behavior of the bubble plume in a stratified environment. In this article, classical plume theory is extended to include gas-water exchange of oxygen and nitrogen as well as bubble expansion due to decreasing pressure during bubble rise. The mathematical plume model is presented in sections 2.1–2.6; then the relationship of the plume model to an overall lake model is discussed briefly in section 2.7. In section 3 the model is applied to the two standard cases of operation, oxygenation and artificial mixing, respectively, and compared with a field experiment. Results and questions remain-

ing open are discussed in section 4, and conclusions are given in section 5.

2. DEVELOPMENT OF THE BUBBLE PLUME MODEL IN STRATIFIED WATER

2.1. Previous Work

Air bubble plumes, induced by the steady release of air into water, have been widely studied both experimentally and theoretically. Previous work, which has been mainly concentrated on plumes in homogeneous water, has revealed various characteristics of the flow: the Gaussian radial distribution of plume velocity and bubbles, the similarity of the radial profiles at different distances from the source, and the entrainment of the surrounding water. Measurements have been carried out over a wide range of water depths from centimeters [Durst *et al.*, 1986] to meters [Kobus, 1973; Ditmars and Cederwall, 1974; Goossens, 1979; Fannelop and Sjoen, 1980; Milgram and Van Houten, 1982] and up to tens of meters [Topham, 1975; Milgram, 1983] with airflow rates from $4 \times 10^{-7} \text{ N m}^3 \text{ s}^{-1}$ (weak plume [Leitch and Baines, 1989]) to $0.66 \text{ N m}^3 \text{ s}^{-1}$ (strong plume [Topham, 1975]).

Theoretical investigations followed the early work of Ditmars and Cederwall [1974], who applied the integral theory of a single-phase plume [Morton *et al.*, 1956] to the two-phase bubble plume. The theory is based on the horizontally integrated equations of conservation of mass, momentum and buoyancy and on the entrainment hypothesis, which states that the volume of entrained water per unit height is proportional to both the local plume velocity and the plume circumference. Compressibility of the air and bubble slipping, i.e., the differential velocity between rising bubbles and the plume water, are two features which are usually also included in bubble plume models. The effect of turbulent transport within the plume was taken into account by Milgram [1983] who introduced the momentum amplification factor γ (ratio of total momentum flux to the momentum flux carried by the mean flow).

Far less work has been done on stratified water bodies.

Fig. 1b. Top view of one diffuser unit, consisting of 54 tubes (courtesy of Ingenierbüro Jungo, Zürich, Switzerland).

McDougall [1978] generalized the integral model for a stratified ambient fluid. Experiments, carried out in strongly stratified [McDougall, 1978] and weakly stratified [Hussian and Narang, 1983] water revealed a double plume structure in which the bubbles remain in the center part, whereas the pure liquid outer part of the plume can spread out at certain levels.

If bubble plumes are applied to destratify lakes and reservoirs it is important to optimize the airflow rates required to overcome a given stratification. A concept of maximum efficiency was demonstrated experimentally for linear stratification by Asaeda and Imberger [1988]. Patterson and Imberger [1989] combined McDougall's [1978] bubble plume model with the one-dimensional dynamic lake model DYRESM to evaluate destratification in response to air bubble plumes and weather. With a similar goal in mind, Zic and Stefan [1990] have even developed models to simulate the entire flow field induced by a bubble plume in a lake.

In this section, the integral plume theory will be extended to include not only bubble expansion [Ditmars and Cederwall, 1974], but also gas exchange (dissolution and stripping) for the dynamics of a buoyant plume. Previous investigators have not included the latter feature, which is important in deep lakes and for weak plumes. The bubble model will also include the effects of density stratification due to gradients of both vertical temperature and dissolved solids. The model can cope with arbitrary ambient profiles of dissolved oxygen and nitrogen.

2.2. Model Features and Assumptions

Previous investigators have presented the basic equations for bubble plumes, and discussed the problem of estimating suitable parameters such as the entrainment coefficient, α , and the spreading ratio of bubbles to fluid flow, λ . The new features of the proposed model for lake use are (1) finite source diameter; (2) induced vertical water velocity at the level of the diffusor as an initial condition; (3) exchange of gases (oxygen and nitrogen primarily) between bubbles and liquid in the plume; (4) conservation equations for specific gases in both dissolved and gas phases; (5) variable gas composition of bubbles during rise; and (6) arbitrary ambient profiles for temperature, salinity (mass of dissolved solids per mass of solution), and dissolved oxygen and nitrogen. Other features included in the plume model are variable bubble radius due to expansion and dissolution; bubble slip velocity and gas mass transfer coefficient as a function of bubble radius, and solubility constants for oxygen and nitrogen as a function of temperature.

The principal assumptions used in developing the model for a steady rising bubble plume in uniform or stratified environment are as follows:

1. The variation of the effective mass density, $\Delta\rho$, is important only in the gravity terms and not for mass or momentum fluxes (Boussinesq assumption, as in all plume theories). This is also equivalent to assuming relative gas volume concentration $V_g \ll 1$.
2. The mass density of gas is neglected compared to that of water in the momentum equation.
3. Uniform distributions ("top hat") across the plume are assumed for water velocity, temperature, dissolved solids, dissolved gases and bubble velocity and concentra-

Fig. 2. Diagram defining the bubble plume geometry, showing circular "top hat" distributions. The top hat radius of the bubble core, λb , is smaller than the plume radius b . The latter is assumed to apply for all properties of dissolved species, including temperature (T) and plume velocity (w). Variables are defined in Tables 1, 3a and 3b.

tion (as well as for individual gases) (Figure 2). This is simpler than trying to use Gaussian profiles, and will yield all the correct scaling relationships; also for large sources whose size is a significant fraction of the depth, we avoid the need for special treatment of a zone of flow establishment, during which top hat profiles at the source would have to change to Gaussian profiles. Morton [1959] and Fannelop and Sjoen [1980] have shown that the top hat distribution captures the main flow features, and that only the constants need to be adjusted.

4. The plume radius b for dissolved species (dissolved solids and dissolved nitrogen and oxygen) and temperature is assumed to be equal to the radius of the top hat plume velocity, whereas that for the bubbles is only λb , where $\lambda \leq 1$. (Although it is expected that dissolved substances will actually spread slightly faster than momentum, such a distinction is an unnecessary complication in this case where the bubbles are the major source of buoyancy.)

5. Ambient currents (other than the plume entrainment velocity) are assumed to be zero.

6. The plume is fully turbulent.

7. The turbulent transport of momentum and scalar quantities (heat, salinity, gases) is neglected compared to advective transport calculated from the mean flow. This is equivalent to taking Milgram's [1983] momentum amplification factor equal to 1.0, which appears reasonable from his experiments.

8. The bubble source is assumed to produce bubbles at a constant rate uniformly distributed over the circular source area of radius λb° .

9. Bubble coalescence is neglected, or else the rate of change of the number of bubbles is prespecified. In this model, the number flux of bubbles, N , is kept constant with height and equal to the number flux at the source.

10. The bubbles are all of uniform size (or are adequately represented by a suspension of equal bubbles). If instead a bubble size spectrum were prespecified, it would be possible

TABLE 1. List of the Eight Basic Plume Variables

Variable	Symbol	Units
Vertical velocity of the plume water	w	m s^{-1}
Plume radius for velocity and dissolved species	b	m
Temperature of plume water and bubbles	T	$^{\circ}\text{C}$
Total dissolved solids in plume water*	$S\rho_w$	kg m^{-3}
Concentration of dissolved O_2	c_{O}	mol m^{-3}
Concentration of dissolved N_2	c_{N}	mol m^{-3}
Concentration of gaseous O_2 †	m_{O}	mol m^{-3}
Concentration of gaseous N_2 †	m_{N}	mol m^{-3}

Variables with index a , i.e., T_a , S_a , $c_{\text{O}a}$, $c_{\text{N}a}$, describe "ambient" (i.e., lake water) properties.

* S denotes salinity (mass of salt per mass of solution).

†Amount of gas per unit bulk volume of the bubble-water mixture within the bubble plume of radius λb .

to replace the number flux of bubbles N by an array $N(r_1, \dots, r_n)$ and to calculate the fate of the individually sized bubbles independently.

11. Water properties of the initial plume are those of lake water at that depth (where the diffuser is installed).

12. Gas exchange between water and bubbles for gases other than oxygen and nitrogen (e.g., argon, carbon dioxide, methane) is neglected. However, other gases could easily be included in the model by adding the relevant conservation equations.

2.3. Conservation Equations

There are eight basic plume variables (Table 1) to be calculated by simultaneous integration of eight differential equations describing the plume dynamics based on the conservation laws for mass, momentum, and heat (Table 2). Several additional parameters are introduced into the model which can be calculated either from the corresponding equations of state (Table 3a) or from empirical approximations taken from the literature (Table 3b). In the following the reader is referred to Tables 1–3 for the definition of variables and parameters.

Continuity for plume water flow. The rate of change of the volume flow is given by the entrainment of ambient water into the plume:

$$\frac{d}{dz} [\pi b^2 w (1 - \lambda^2 V_g)] = 2\alpha \pi b w \quad (\text{m}^2 \text{s}^{-1}) \quad (1)$$

where z is the vertical coordinate (positive upward) and α , the entrainment coefficient for the top hat plume, is $\sqrt{2}$ times the corresponding value used for a Gaussian plume model, α_g , i.e.,

$$\alpha = \sqrt{2} \alpha_g \quad (\text{dimensionless}) \quad (2)$$

as explained by Morton [1959]. Furthermore, the top hat radius is

$$b = \sqrt{2} b_g \quad (\text{m}) \quad (3)$$

where b_g is the plume width in the Gaussian model ($w = w_m \exp(-r^2/b_g^2)$). The term $\lambda^2 V_g$ is a correction for the volume occupied by the gas bubbles. Since $V_g < 10^{-2}$ for bubble plumes of interest for lake restoration, and following the Boussinesq assumption, we may rewrite the equation without the V_g term:

$$\frac{d}{dz} (\pi b^2 w) = 2\alpha \pi b w \quad (\text{m}^2 \text{s}^{-1}) \quad (4)$$

Momentum flux. We may again neglect the gas volume and gas momentum in calculating the momentum flux $M = \pi b^2 w^2$ of the plume. The rate of change of M with height z is equal to the buoyant force per unit height of the plume:

$$\begin{aligned} \frac{d}{dz} (\pi b^2 w^2) &= \frac{\rho_a - \rho_p}{\rho_p} g \pi b^2 \lambda^2 \\ &+ \frac{\rho_a - \rho_w}{\rho_p} g \pi b^2 (1 - \lambda^2) \quad (\text{m}^3 \text{s}^{-2}) \end{aligned} \quad (5)$$

(ρ_a and ρ_w are densities of ambient and plume water, respectively, and ρ_p is the density of the plume bubble-water mixture). On the right-hand side the first term gives the buoyant force for the area of the bubble distribution top hat, $\pi \lambda^2 b^2$. The second term is the (negative) buoyancy for the fluid inside the annulus (Figure 2) between the bubble top hat (radius λb) and the plume velocity top hat (radius b). When there is no ambient fluid stratification ($\rho_a = \rho_w$) or when $\lambda \approx 1$, the second term vanishes.

Heat or temperature flux. Assuming that the coefficient

TABLE 2. Definition of the Eight Integrated Flux Variables

Variable	Formula	Units
Volume flux of plume water	$\mu = \pi b^2 w$	$\text{m}^3 \text{s}^{-1}$
Momentum flux	$M = \pi b^2 w^2$	$\text{m}^4 \text{s}^{-2}$
Temperature flux (relative to $T = 0^{\circ}\text{C}$)	$F_T = \mu T$	$^{\circ}\text{C m}^3 \text{s}^{-1}$
Flux of total dissolved solids	$F_S = \mu S \rho_w$	kg s^{-1}
Flux of dissolved O_2 , N_2	$D_i = \mu c_i$; $i = \text{O}, \text{N}$	mol s^{-1}
Flux of gaseous O_2 , N_2	$G_i = \pi b^2 \lambda^2 (w + w_b) m_i$; $i = \text{O}, \text{N}$	mol s^{-1}

TABLE 3a. Equations of State

Parameter	Equation	Comments
Partial pressures within bubble at depth z	$p_i = [m_i/(m_O + m_N)]p = [G_i/(G_O + G_N)]p$ for $i = O, N$ (bar)	Total pressure p is given by the hydrostatic approximation
Total pressure	$p = p_s + a \int_z^{z_s} \rho_a g dz = p_s + a \bar{\rho}_a g(z_s - z)$ (bar)	p_s is atmospheric pressure at lake surface (bar); z is vertical distance above diffusor (m); z_s is height of lake surface above diffusor (m); g is acceleration of gravity, equal to 9.81 m s^{-2} ; $\bar{\rho}_a$ is mean density of ambient water; and a is a pressure conversion factor (pascals to bars), equal to $10^{-5} \text{ bar kg}^{-1} \text{ m s}^2$.
Water density at atmospheric pressure	$\rho_a = f_T(T_a) + f_S(S_a)$ for ambient lake water, $\rho_w = f_T(T) + f_S(S)$ for plume water	$f_T(T) = 999.868 \text{ kg m}^{-3} + 10^{-3} [a_1 T + a_2 T^2 + a_3 T^3]$; $a_1 = 65.185 \text{ kg m}^{-3} \text{ }^\circ\text{C}^{-1}$; $a_2 = -8.4878 \text{ kg m}^{-3} \text{ }^\circ\text{C}^{-2}$; $a_3 = 0.05607 \text{ kg m}^{-3} \text{ }^\circ\text{C}^{-3}$; $f_S(S) = \gamma S$, where either $\gamma = 0.802 \text{ kg m}^{-3} (\text{‰})^{-1}$ if S is salinity expressed as dissolved salt content in per mil or $\gamma = 0.705 \times 10^{-3} \text{ kg m}^{-3} (\mu\text{S/cm})^{-1}$ if S is expressed in terms of the electrical conductivity measured at 20°C (modified after Bührer and Ambühl [1975]).
Density of bubble-water mixture in plume	$\rho_p = (1 - V_g) \rho_w$	
Gas volume*		
Ideal gas law	$V_g = [(m_O + m_N)/p]RT$	$R = 8.314 \text{ J mol}^{-1} \text{ K}^{-1}$ (gas constant) and T is absolute temperature of plume water (K)
Van der Waals gas law	Implicit equation for V_g : $[V_g/(m_O + m_N) - b_v] \cdot [p + a_v(m_O + m_N)^2/V_g^2] = RT$	$a_v = 1.4 \times 10^{-6} \text{ bar m}^6 \text{ mol}^{-1}$; $b_v = 32 \times 10^{-6} \text{ m}^3 \text{ mol}^{-1}$
Bubble radius	$r = (3V_g/\lambda^2 b^2(w + w_b/4\pi N))^{1/3}$	N is the number flux of bubbles released at diffusor per unit time

* V_g is the gas volume per total volume of the bubble-water mixture in the inner core of the plume (dimensionless).

of specific heat per unit water volume ($\rho_w \kappa$) is constant, and neglecting the heat content of the gas volume, the conservation equation for the heat flux ($\pi b^2 w \rho_w \kappa T$) can be reduced to the corresponding expression for the temperature flux $F_T = \pi b^2 w T$:

$$\frac{d}{dz} (\pi b^2 w T) = 2\alpha \pi b w T_a \quad (^\circ\text{C m}^2 \text{ s}^{-1}) \quad (6)$$

where T_a is the ambient water temperature given by measured temperature profiles or by the lake model.

Flux of total dissolved solids. For water temperatures close to 4°C , the concentration of dissolved solids, ex-

pressed by the salinity S , plays a role in establishing the density stratification. Hence it is necessary to include the mass balance for dissolved solids:

$$\frac{d}{dz} (\pi b^2 w \rho_w S) = 2\alpha \pi b w \rho_a S_a \quad (\text{kg m}^{-1} \text{ s}^{-1}) \quad (7)$$

where S_a is the ambient salinity.

Fluxes of dissolved oxygen and nitrogen. There are two gases of interest, molecular oxygen (O_2) and nitrogen (N_2), and two phases, dissolved and gaseous, for each. For each species the gas transfer term appears as a gain (or loss) in the dissolved gas conservation equation and as a loss (or gain) in

TABLE 3b. Parameter Approximations

Parameter	Approximation	Units
Bubble slip velocity (Figure 5)	$w_b(r) = w_1(r/r_*)^{1.357}$ $r/r_* < 7.0 \times 10^{-4}$ $w_b(r) = w_2$ $7.0 \times 10^{-4} < r/r_* < 5.1 \times 10^{-3}$ $w_b(r) = w_3(r/r_*)^{0.547}$ $r/r_* > 5.1 \times 10^{-3}$ with $w_1 = 4474 \text{ m s}^{-1}$; $w_2 = 0.23 \text{ m s}^{-1}$; $w_3 = 4.202 \text{ m s}^{-1}$; $r_* = 1 \text{ m}$	m s^{-1} m s^{-1} m s^{-1}
Gas transfer coefficient for molecular oxygen and nitrogen (Figure 3)	$\beta_i(r) = 0.6 \text{ m s}^{-1} (r/r_*)$ $(r/r_*) < 6.67 \times 10^{-4}$, $i = O, N$ $\beta_i(r) = 4 \times 10^{-4} \text{ m s}^{-1}$ $(r/r_*) > 6.67 \times 10^{-4}$, $i = O, N$	m s^{-1} m s^{-1}
Molecular oxygen solubility constant (Figure 4)	$K_O(T) = K_{O1} + K_{O2}T + K_{O3}T^2$ with $K_{O1} = 2.125 \text{ mol m}^{-3} \text{ bar}^{-1}$; $K_{O2} = -0.05021 \text{ mol m}^{-3} \text{ bar}^{-1} \text{ }^\circ\text{C}^{-1}$; $K_{O3} = 5.77 \times 10^{-4} \text{ mol m}^{-3} \text{ bar}^{-1} \text{ }^\circ\text{C}^{-2}$	$\text{mol m}^{-3} \text{ bar}^{-1}$
Molecular nitrogen solubility constant (Figure 4)	$K_N(T) = K_{N1} + K_{N2}T + K_{N3}T^2$ with $K_{N1} = 1.042 \text{ mol m}^{-3} \text{ bar}^{-1}$; $K_{N2} = -0.0245 \text{ mol m}^{-3} \text{ bar}^{-1} \text{ }^\circ\text{C}^{-1}$; $K_{N3} = 3.171 \times 10^{-4} \text{ mol m}^{-3} \text{ bar}^{-1} \text{ }^\circ\text{C}^{-2}$	$\text{mol m}^{-3} \text{ bar}^{-1}$
Constants		
Entrainment coefficient	$\alpha = 0.11$	
Plume radius ratio	$\lambda = 0.8$

Fig. 3. Mass transfer coefficients β_O for oxygen measured in tap water as a function of bubble radius, based on data (solid circles) compiled by Motarjemi and Jameson [1978]. In this model the same mass transfer coefficients for oxygen and nitrogen were used, since the two molecules have nearly identical molecular diffusivities. The solid line shows the parametrization $\beta_O(r) = \beta_N(r)$ used in this paper.

the gas phase conservation equation. The gas flux through bubble surfaces is given by

$$\beta_i(c_{si} - c_i) \quad (\text{mol m}^{-2} \text{ s}^{-1}) \quad (8)$$

where β_i is the mass transfer coefficient (Figure 3) for the gas species i (O_2 or N_2), c_i is the in situ dissolved gas concentration, and c_{si} is the saturation concentration determined by Henry's law:

$$c_{si} = K_i p_i \quad (\text{mol m}^{-3}) \quad (9)$$

Here p_i is the in situ partial pressure of the gas phase of species i and K_i is the solubility constant (Figure 4).

In order to calculate the total gas transfer per unit height of the bubble plume we use the total number flux of bubbles, N , a quantity which, according to the list of assumptions (see section 2.2, ninth assumption) is determined by the initial conditions (initial bubble radius r^0 , total gas input at diffusor and diffusor depth; Tables 3a and 4) and then remains constant within the plume. The number of bubbles per unit plume height is then simply $N/(w + w_b)$, where $(w + w_b)$ is the total bubble velocity (w_b is bubble slip velocity; Figure 5). The total surface area of bubbles per unit plume height is found by multiplying by the area of each bubble, i.e., $4\pi r^2 N/(w + w_b)$. The total gas transfer rate per unit height is then obtained by multiplying this aggregate surface area (per unit height) by the local gas flux:

$$\frac{4\pi r^2 N}{w + w_b} \beta_i(K_i p_i - c_i) \quad (\text{mol s}^{-1} \text{ m}^{-1}) \quad (10)$$

The conservation equation for each dissolved species i , i.e., O_2 and N_2 , includes entrainment of ambient lake water (concentrations c_{ia} for $i = \text{O}$ and N) and gas exchange:

$$\frac{d}{dz} (\pi b^2 w c_O) = 2\alpha \pi b w c_{Oa} + \frac{4\pi r^2 N}{w + w_b} \beta_O(K_O p_O - c_O) \quad (\text{mol s}^{-1} \text{ m}^{-1}) \quad (11)$$

$$\frac{d}{dz} (\pi b^2 w c_N) = 2\alpha \pi b w c_{Na} + \frac{4\pi r^2 N}{w + w_b} \beta_N(K_N p_N - c_N) \quad (\text{mol s}^{-1} \text{ m}^{-1}) \quad (12)$$

Fluxes of gaseous oxygen and nitrogen. The rate of change of advective flux of gaseous O_2 and N_2 is equal to the rate of dissolution or stripping as determined above:

$$\frac{d}{dz} [\pi b^2 \lambda^2 (w + w_b) m_O] = -\frac{4\pi r^2 N}{w + w_b} \beta_O(K_O p_O - c_O) \quad (\text{mol s}^{-1} \text{ m}^{-1}) \quad (13)$$

$$\frac{d}{dz} [\pi b^2 \lambda^2 (w + w_b) m_N] = -\frac{4\pi r^2 N}{w + w_b} \beta_N(K_N p_N - c_N) \quad (\text{mol s}^{-1} \text{ m}^{-1}) \quad (14)$$

(m_O and m_N are the gaseous concentration of the species per unit bulk volume of the bubble-water mixture). Note that the relevant volume flux for the gaseous components contains the total bubble velocity, $w + w_b$, and the reduced plume radius, λb . Upon solving these equations it is found that the proportion of O_2 and N_2 in the rising gas bubbles is changing. For instance, pure O_2 bubbles will strip N_2 from the water column at the same time that O_2 is being dissolved.

The assumptions made in developing these eight equations could be generalized, but at the price of adding parameters. For instance, one could distinguish five separate top hat radii for velocity, bubbles, dissolved gases, temperature, and salinity (giving four different λ ratios). The refinement is not warranted at this time, since information is lacking on how to select different λ values. Hence, we use only two top hat radii: b for velocity, temperature, salinity, and dissolved gases, and λb for bubbles. It is well known from previous investigations that λ is significantly less than 1, and probably about 0.8 [Milgram, 1983].

2.4. Variable Transformation

To simplify the solution we let the basic dependent variables be the eight integrated fluxes defined in Table 2, namely those for water volume, momentum, temperature, dissolved solids and the dissolved and gaseous phases of molecular O_2 and N_2 . With these definitions, the eight conservation equations ((4)–(7) and (11)–(14)) may be rewritten in terms of the new variables:

$$\frac{d\mu}{dz} = 2\alpha(\pi M)^{1/2} \quad (15)$$

Fig. 4. Solubility constants K_i of molecular nitrogen and oxygen as functions of temperature. Dots represent values taken from Marshall [1976]. Solid lines show second-order polynomial fits used in this model.

TABLE 4. Initial Conditions for Bubble Plume Calculation

Initial Plume Variable	Symbol	Method for Determining Initial Value
Plume radius	b°	effective radius of the entire diffuser unit (diffuser geometry)
Vertical plume velocity	w°	w° depends on source and plume properties; see section 2.5
Volume flux of plume water	μ°	$\mu^\circ = \pi b^{\circ 2} w^\circ$
Momentum flux	M°	$M^\circ = \pi b^{\circ 2} w^{\circ 2}$
Temperature	T°	ambient temperature at diffuser depth
Temperature flux	F_T°	$F_T^\circ = \mu^\circ T^\circ$
Dissolved species		
Concentrations	$S^\circ, c_O^\circ, c_N^\circ$	ambient concentrations at diffuser depth
Fluxes	$F_S^\circ, D_O^\circ, D_N^\circ$	$F_S^\circ = \mu^\circ S^\circ, D_O^\circ = \mu^\circ c_O^\circ, D_N^\circ = \mu^\circ c_N^\circ$
Gas fluxes	G_N°, G_O°	nitrogen and/or oxygen input rate (mode of operation)
Bubble radius*	r°	design (material and porosity) of diffuser
Pressure	p°	depth of diffuser (topography and installation) and ambient density profile
Number flux of bubbles†	N°	initial gas fluxes, bubble radius and pressure at the diffuser depth (Table 3a)

*It may be possible to vary bubble radius during operation by switching between different installed diffuser tubes. For the system "Tanytarsus" a switch between small (oxygenation mode) and large (artificial mixing mode) bubbles is possible.

†The number flux of bubbles, $N(z)$, is kept constant in the plume: $N(z) = N^\circ$.

$$\frac{dM}{dz} = \frac{\rho_a - \rho_p}{\rho_p} g \lambda^2 \frac{\mu^2}{M} + \frac{\rho_a - \rho_w}{\rho_p} g \frac{\mu^2}{M} (1 - \lambda^2) \quad (16)$$

$$\frac{dF_T}{dz} = 2\alpha(\pi M)^{1/2} T_a \quad (17)$$

$$\frac{dF_S}{dz} = 2\alpha(\pi M)^{1/2} \rho_a S_a \quad (18)$$

$$\frac{dD_O}{dz} = 2\alpha(\pi M)^{1/2} c_{Oa} + \frac{4\pi r^2 N}{(M/\mu) + w_b} \beta_O \left(K_O p_O - \frac{D_O}{\mu} \right) \quad (19)$$

$$\frac{dD_N}{dz} = 2\alpha(\pi M)^{1/2} c_{Na} + \frac{4\pi r^2 N}{(M/\mu) + w_b} \beta_N \left(K_N p_N - \frac{D_N}{\mu} \right) \quad (20)$$

$$\frac{dG_O}{dz} = -\frac{4\pi r^2 N}{(M/\mu) + w_b} \beta_O \left(K_O p_O - \frac{D_O}{\mu} \right) \quad (21)$$

$$\frac{dG_N}{dz} = -\frac{4\pi r^2 N}{(M/\mu) + w_b} \beta_N \left(K_N p_N - \frac{D_N}{\mu} \right) \quad (22)$$

This is a set of eight coupled nonlinear first-order differential equations which can be integrated numerically. The system contains five additional variables ($\rho_p, \rho_w, r, p_O, p_N$) which are linked to basic variables by equations of state (Table 3a) and thus can be adjusted after every integration step. Furthermore, there are other physicochemical parameters which are either kept constant throughout the model calculation (e.g., relative bubble plume radius λ , entrainment coefficient α) or can be calculated from empirical approximations (Table 3b). The number flux of bubbles N is established by the initial gas volume flux and the assumed initial bubble radius, and then remains constant within the plume.

2.5. Initial Conditions

The model calculations depend crucially on both the initial plume conditions, which are summarized in Table 4, and the ambient lake water profiles. Whereas the ambient profiles are given by the state of the lake (influenced however by the plume operation in the long term, see section 2.7), the initial conditions can be controlled to a certain extent by the design and operation of the diffuser units.

We have to specify the initial values of the eight flux variables defined in Table 2. According to the eleventh assumption, the initial values of temperature T° , salinity S° , dissolved O_2 concentration c_O° and dissolved N_2 concentra-

Fig. 5. Terminal slip velocity of air bubbles, w_b , measured in tap water (solid circles) and distilled water (open circles) [Haberman and Morton, 1954]. The solid line represents the approximation used in this model.

tion c_N^0 are given by the ambient lake water characteristics at the depth of the diffuser. However, to determine the respective initial fluxes we need to multiply each by the initial volume flux μ^0 which is discussed below. The gaseous O_2 and N_2 fluxes, G_O^0 and G_N^0 , are the key initial variables to be controlled by the mode of operation of the diffuser. The initial bubble radius, r^0 , is determined mainly by the design of the diffuser (pore size) and cannot be controlled by varying the gas flow rate (unless it is very large). The number flux of bubbles, $N = N^0$, kept constant in the whole plume, is determined by total gas flux, $G_O^0 + G_N^0$, initial bubble radius, r^0 , and pressure, p^0 (Table 3a), and is not an independent initial condition.

The initial inner plume radius (λb^0) is determined by the apparent size of the bubble source area. Specifying λ and b^0 establishes the width of the velocity plume. In practice, a diffuser unit may consist of several diffuser tubes distributed in a starlike pattern of six arms, as shown in Figure 1b [Stadelmann, 1988]. In such cases we consider the area of the entire diffuser unit as a uniform source of buoyancy and consequently set the initial plume area $\pi(\lambda b^0)^2$ equal to the circular area covered by the tubes.

Mathematically, the vertical velocity w^0 is an additional independent initial variable. In reality, however, the initial water velocity should be determined by the properties of the source and of the ambient water. It has been found useful to define a densimetric Froude number (dimensionless) as

$$Fr = \frac{w}{[2\lambda b g(\rho_a - \rho_p)/\rho_p]^{1/2}} \quad (23)$$

For a single-phase plume Fr is simply inversely proportional to the local Richardson number Ri [Fischer et al., 1979] which has the unique value of 0.557 along the plume, except near the source. The constant of proportionality depends on the type of plume profiles being used; here $Fr = (\pi/4)^{1/4} Ri^{-1}$. Thus the Froude number of the plume should everywhere have a value of 1.69, except near the source. However, for bubble plumes there is an additional characteristic velocity, the bubble slip velocity w_b . Thus we do not expect a unique starting condition to exist. For very small bubbles ($w_b \rightarrow 0$), the initial Froude number should be the same as for a single-phase plume. From the model calculations that follow we find that as z increases, Fr does indeed tend to a nearly constant value of about 1.7. For initial values of $Fr^0 < 1.5$, the plume water velocity w initially (immediately above the diffuser) increases with increasing z (reasonable behavior), while for $Fr^0 > 1.5$, w first decreases in the model (unlikely). Therefore, for the standard case (section 3.1) we use the value in between, namely, $Fr^0 = 1.6$, and for the sensitivity analysis (section 3.2) we analyze the effect of varying Fr^0 values.

2.6. Solution Procedure

The stationary solution for the flux variables (Table 2) is calculated by a three-step iteration loop. In the first step the eight differential flux equations ((15)–(22)) are integrated using the standard first-order Euler method, starting from the initial values listed in Table 4. The vertical coordinate increment dz is controlled by keeping the maximum relative change of the flux variables below a chosen value (1%). By sensitivity analysis it was found that the 1% criterion is sufficiently small to guarantee that the final values of the

plume variables (Table 1) are not increment dependent ($<1\%$ relative error). After each iteration the equations of state (Table 3a) are solved for the partial pressure of O_2 and N_2 (p_O , p_N), the water and plume densities (ρ_w , ρ_p), and the bubble radius (r). In the third and last step, the remaining parameters, bubble slip velocity, w_b (Figure 5), mass transfer coefficients, β_i (Figure 3), and the solubility constants, K_O and K_N (Figure 4), are calculated from the parameterization listed in Table 3b.

The iteration is interrupted and the program stopped when the plume velocity approaches zero or when the plume reaches the surface. Comparison of the final plume water density with the ambient density profile then allows the determination of the equilibrium depth, i.e., the depth to which the plume water would sink back if no further mixing occurred. However, the fallback plume (which we do not attempt to model) entrains some additional water on the way down; as a result, in reality it does not fall back completely to the calculated equilibrium depth.

2.7. Plume Model in Relation to Lake Model

The plume model is only one component of the overall lake model. In the simplest case, the lake is considered to consist of two parts: the small plume zone (one or more vertical columns) and the much larger lake zone outside the plumes (Figure 1a). The lake zone may in many cases be considered to be horizontally well mixed or uniform in properties (such as temperature T or dissolved O_2); i.e., $T_a(x, y, z, t) = T_a(z, t)$.

The plume is assumed to be in quasi-steady state, operating for some period of time Δt (of the order of days) with essentially fixed ambient environmental profiles determined by the lake zone. Then the lake zone profiles are adjusted to reflect the results of the plume action over the given time period Δt . In other words, the ambient profiles change so slowly that they may be considered piecewise constant for the purpose of the plume calculations.

The presentation of the lake zone model is not the purpose of this paper. However, for just the physical aspects, the basic equations for coupling to the plume model are conservation of water mass and conservation of any scalar species or pollutant. For conservation of water mass at level z , the vertical rate of change of the vertical flow, wA , through the lake cross section A (w is far-field vertical upward velocity, Figure 1a), is equal to the plume outflow per unit height $q_p(z)$ minus the plume entrainment $q_e(z)$:

$$\frac{d(wA)}{dz} = q_p - q_e \quad (\text{m}^2 \text{ s}^{-1}) \quad (24)$$

By definition, either one or the other of q_p and q_e must be zero at a particular level z ; i.e., the plume is either entraining ($q_p = 0$) or detraining ($q_e = 0$).

In the far field of the lake the mass balance of any conservative substance with concentration $c_a(z)$ is given by

$$A \frac{\partial c_a}{\partial t} + \frac{\partial(c_a w A)}{\partial z} = q_p c_p - q_e c_a \quad (\text{mol s}^{-1} \text{ m}^{-1}) \quad (25)$$

where c_a and c_p describe ambient and mean plume concentrations, respectively.

The plume model uses $c_a(z)$ (and $T_a(z)$) as inputs, and produces q_p , q_e , and c_p as outputs (Figure 1a) which are used to compute new values of w and c_p at given time

TABLE 5. Characteristics of Internal Lake Restoration Measures in Baldeggersee (Switzerland)

Parameter	Value or Description
<i>Lake Data</i>	
Location	47°12'N, 8°16'E, (15 km north of Lucerne, Switzerland)
Surface altitude	462 m above sea level
Surface area	5.2 km ²
Volume	176 × 10 ⁶ m ³
Maximum depth	66 m
<i>Oxygenation Mode (May–October)</i>	
Operation	four to six diffusers at depth between 42 and 64 m
Total O ₂ input	3–4.5 × 10 ³ kg d ⁻¹
O ₂ input per diffuser unit	0.025–0.037 N m ³ s ⁻¹ *
<i>Artificial Mixing Mode (November–April)</i>	
Operation	three to four diffusers or nozzles at depth between 58 and 64 m
Total air input	0.056 N m ³ s ⁻¹
Air input per diffuser unit	0.014–0.017 N m ³ s ⁻¹

*1 N m³ = 1 m³ of gas at 1 bar and 0°C.

intervals Δt by finite difference equations representing the lake model (equations (24) and (25)). This procedure was demonstrated by Imboden [1985].

3. THE CASE OF BALDEGGERSEE: APPLICATION AND SENSITIVITY EVALUATION OF THE MODEL

In this section application of the model to Baldeggersee, a highly eutrophic lake (Table 5), is described. Among all

Swiss lakes, Baldeggersee has the longest record of artificial oxygen and air input [Imboden, 1985]. Artificial destratification with an air bubble plume began in February 1982 followed by a first experimental oxygenation of the hypolimnion using pure oxygen during the next summer.

Vertical profiles of O₂, temperature and salinity measured in July 1983 (oxygenation) and November 1983 (artificial mixing) are chosen as the input data to run the bubble plume model (Figure 6). Measured N₂ profiles in Baldeggersee were approximately constant and close to the surface saturation value in winter. Hence a constant value of $c_N = 0.71$ mol m⁻³ (20 g m⁻³), corresponding to N₂ saturation at $T = 4^\circ\text{C}$ at the altitude of Baldeggersee, was taken as the ambient N₂ concentration. It is assumed that the density of the water is defined only by the temperature and the concentration of dissolved solids, which consist mainly of Ca²⁺ and HCO₃⁻. The influence of dissolved solids on the vertical density gradient is negligible in the thermocline of most lakes, although in the prerestoration period chemical density gradients in the deep hypolimnion may be important as well [Joller, 1985]. The ambient density profiles for July and November 1983 shown in Figure 6d were calculated from water temperature and electrical conductivity using the modified equations given by Bührer and Ambühl [1975] (Table 3a).

3.1. The Standard Cases

The model calculations are based on two standard situations as listed in Table 6. Since the initial values of plume size, vertical plume velocity and bubble radius are not exactly known, reasonable values for the standard cases

Fig. 6. Temperature, salinity, density ($\sigma = \rho - 1000$ kg m⁻³), and oxygen profiles, measured in July (solid squares) and November (open circles) 1983 in Baldeggersee. These profiles serve as input data for the two standard case model calculations.

TABLE 6. Model Input Values for One Diffuser Unit, Applied to Standard Case and Sensitivity Analysis

Parameter		Standard Case	Sensitivity Analysis
Diffuser depth, m	z^o	65	n.v.*
Initial plume size, m ²	πb^o^2	20	0–50
Entrainment factor	α	0.11	0.05–0.2
<i>Oxygenation</i>			
O ₂ input, † N m ³ s ⁻¹		0.0062	0.001–0.1
Initial Froude number ‡	Fr^o	1.6 ($w^0 = 0.125$ m/s)	1.0–2.0
Initial bubble radius, m	r^o	0.001	10 ⁻⁴ to 10 ⁻¹
Temperature, salinity and oxygen profiles		July 1983 (Figure 6)	n.v.
<i>Artificial Mixing</i>			
Air input, † N m ³ /s		0.014	n.v.
Initial Froude number ‡	Fr^o	1.6 ($w^0 = 0.172$ m/s)	0.5–2.5
Initial bubble radius, m	r^o	0.006	n.v.
Temperature, salinity and oxygen profiles		Nov. 1983 (Figure 6)	n.v.

*Here n.v. denotes not varied.

†1 N m³ = 1 m³ of gas at 1 bar and 0°C.

‡See section 2.5 for the choice of the initial Froude number, equation (23).

were selected followed by a sensitivity analysis of the model results with respect to these choices. The calculated vertical variations of important model variables along the plume are shown in Figure 7 for both modes of operation, i.e., pure oxygen input during the summer and air input during the winter.

The most important difference between the two cases (Table 6) lies in the initial bubble radius, i.e., 1 mm in July versus 6 mm in November. During oxygenation (July) the

surface-to-volume ratio of the bubbles is large. The bubbles dissolve quickly and their volume is only 0.1% of the initial value at about 40 m depth (Figure 7c). When the bubbles disappear, the colder and heavier water from the lake bottom loses its buoyancy (Figure 7e), the plume stops (Figure 7a), disperses, sinks part way, and merges with the surrounding lake water at the depth of neutral density. The model calculation is restricted to the zone where the plume is still rising. At the depth of maximum plume rise (DMPR) some

Fig. 7. Model calculations for bubble plumes in Baldeggersee using standard parameters from Table 6 and ambient profiles from Figure 6. Profiles of important model variables along the vertical plume axis are shown for the two modes of operation: oxygenation during summer (July) and artificial mixing with compressed air during winter (November).

Fig. 8. Sensitivity of plume model calculated for the case of oxygen input into Baldeggersee using the standard situation of July 1983 (Table 6). Only one parameter is varied from its standard value (labeled by "STD") for each calculation. See text for definition of equilibrium depth.

bubbles may still exist if the system operates with large enough bubbles. Possible further dissolution or secondary plume creation [McDougall, 1978] by the escaping bubbles above DMPR is not considered in this model. If small bubbles are used, however, they disappear before the DMPR is reached. In the momentum overshoot phase of negative buoyancy no secondary plumes have to be considered; the plume then behaves as an ordinary one [Morton et al., 1956].

In contrast, during artificial mixing with air (November), gas exchange is slower since the bubbles have a larger initial radius. The rate of increase in bubble size due to decompression is similar to the rate of decrease due to dissolution. The net effect, however, is a slight increase in the in situ bubble radius (Figure 7c). Therefore, the plume retains its buoyancy up to 20 m below the lake surface (Figure 7f). The remaining momentum brings the plume to the surface (Figures 7a and 7b).

Note that the differences between the summer and winter cases are primarily related to the different initial bubble sizes; the chemical composition of the bubbles (oxygen or air, respectively) is less important for deep lakes since the high hydrostatic in situ pressure makes the gas exchange rate nearly independent of the ambient dissolved gas concentration at this depth.

The larger size of the bubbles is not the only reason why, in the artificial mixing case, the plume reaches the surface. Higher gas flux and weaker ambient stratification also support stronger plumes. Higher gas flux leads to a larger density difference (i.e., buoyancy), and a higher initial plume velocity (momentum flux). The weak stratification of the water column provides little negative buoyancy to stop the

plume. The effects of initial gas flux and bubble size on the plume will be investigated in the following sensitivity analysis.

3.2. Sensitivity Analysis

With respect to the application of the model to internal lake restoration the most important model characteristics are the vertical extension of the plume, the magnitude of the induced vertical volume flux of water, and the fraction of oxygen transferred to the dissolved phase. It is therefore important to investigate the sensitivity of these quantities with respect to those input parameters which either can be influenced by the design of the diffuser system or are not well known from field observations. It is not easy to compare different model results for the winter case in a didactically clear way, since, depending on the parameters, the modeled plumes may or may not reach the surface. Therefore the following sensitivity analysis refers only to the summer case, which allows easier comparison (Figure 8). For each calculation, all parameters except the one which is varied correspond to the standard case (Table 6).

Among all the parameters, initial bubble size and initial gas flow have the largest influence on the behavior of the bubble plume, especially on the depth of maximum plume rise (DMPR). If the initial bubble radius is less than 1 mm, the DMPR is about 25–28 m and is nearly independent of bubble size (Figure 8a). For larger bubbles, the DMPR becomes shallower and the plume reaches its highest level for a radius of about 1 cm. The curve (Figure 8a) can be qualitatively explained in terms of the size dependence of the

Fig. 9. Maximum plume rise calculated as a function of initial bubble size for the standard situation of July 1983 with and without gas exchange (i.e., dissolution of bubbles). Plume dynamics is strongly dependent on dissolution for bubbles of radius smaller than about 5 mm.

gas exchange and bubble slip velocity alone. Since the gas exchange rate (fractional volume changes per unit time) depends on the surface-to-volume ratio of the bubbles, the oxygen dissolves faster in the lake if the bubbles are small. In addition, smaller bubbles have lower slip velocities and thus a longer contact time with the water before reaching the DMPR.

For bubbles with a radius of less than about 0.7 mm, the exchange rate (fractional volume changes per unit time) is approximately independent of bubble size, since the transfer velocity is proportional to bubble radius, whereas the surface-to-volume ratio is inversely proportional to bubble radius (Table 3b and Figure 3). As a consequence, the smaller and slower the bubbles are, the more buoyancy is generated and the longer the buoyancy lasts. This causes the plume to rise higher. Bubbles with a radius greater than about 0.7 mm have decreasing gas exchange rates per unit volume (since the transfer rate is constant) and constant slip velocities up to a radius of 5 mm. The height of the plume will consequently increase with bubble size. For bubbles with a radius greater than about 10 mm, the height of the plume decreases again since dissolution no longer has any influence (Figure 9). The bubbles rise so rapidly that the buoyancy (proportional to $(w + w_b)^{-1}$) is significantly diminished; the bubbles escape the DMPR and start a new plume (not described by the model).

Because pure oxygen is expensive it is of major importance to operate a lake oxygenation system so as to minimize the amount of oxygen lost by escaping bubbles and to deliver the oxygen as exactly as possible to the required depths. As shown in Figure 8b, for bubbles with a radius of less than 3 mm, more than 99% of the O_2 dissolves. For bubbles with a radius of 3 mm, vertical water circulation reaches a maximum while no O_2 bubbles escape from the hypolimnion. For larger bubbles the total transfer rate decreases rapidly to reach a value of less than 10% for a radius greater than about 3 cm.

The oxygen input rate directly influences the density and thus the buoyancy of the plume. As seen in Figure 8c, an increase in the O_2 input rate from 10^{-3} to $10^{-1} \text{ N m}^3 \text{ s}^{-1}$ causes the DMPR to rise from 45 m to 10 m depth. For large O_2 input rates the DMPR is determined mainly by the position of the thermocline, i.e., the zone of maximum density gradient. If the total oxygen flow rate and the plume

intrusion depth range are prescribed, the model allows the determination of the minimum number of diffuser units needed to reoxygenate at the desired depths. For example, if the oxygen input per diffuser is reduced to 50% (e.g., by doubling the number of diffusers in the lake), the DMPR sinks by 8 m (Figure 8c). Likewise, for artificial mixing with air bubbles, the model can be used to find the optimal bubble size and the minimum airflow rate as a function of the number of diffusers, provided that the ambient stratification and the turnover rate are given.

The initial plume size and initial Froude number (or water velocity), two parameters which are difficult to estimate without additional and complicated field measurements, do not have a significant influence on the DMPR (Figures 8d and 8e). The initial plume size cannot vary more than between about half the diffuser area (10 m^2) and twice this area (40 m^2). The initial plume velocity is expected to correspond approximately to a densimetric Froude number of 1.6, which we choose slightly smaller than the apparent asymptotic value in the near field of 1.7. The low sensitivity of the model calculation with respect to these two parameters means that during oxygenation the properties of the plume are determined mainly by the plume's dynamics and not by the initial conditions.

As mentioned above, the entrainment factor is an empirical constant which has been determined in several laboratory experiments [Milgram, 1983]. A typical value for Gaussian profiles is 0.08, which is equivalent to 0.11 for top hat profiles. Since the entrainment of surrounding water decelerates the water plume, the DMPR depends strongly on this factor (Figure 8f).

Bubble dissolution has a drastic effect on plume dynamics. In Figure 9 the calculated DMPR is shown as a function of initial bubble size with and without gas exchange, respectively. Since small bubbles dissolve faster and have lower slip velocities than large ones, the effect of dissolution on the DMPR is most prominent for small bubbles. For radii larger than about 2 cm, dissolution can be ignored. The result dramatically shows the importance of gas exchange for bubble plume models, if the initial bubble radius is less than about 5 mm, in deep water (65 m depth in this example).

3.3. Artificial Tracer Experiment

On August 22/23, 1983, in collaboration with the Institute of Geography of the University of Bern, a tracer experiment was conducted in Baldeggersee. During a period of 10 min, 1.5 kg of uranin, a fluorescent dye which had previously been dissolved in 200 L of water taken from the lake bottom, was introduced into the plume at a depth of 57 m, just above one of the diffusers, which at this time was releasing $0.0083 \text{ N m}^3 \text{ s}^{-1}$ of O_2 . The tracer was pumped from a platform into a hose fixed to the diffuser unit. The temporal evolution of the dye cloud was then measured using an in situ fluorometer operated from the platform.

During the initial phase of the experiment, part of the dye rose up to a depth of 5 m. The average rising velocity was 0.16 m s^{-1} below 40 m depth, and about 0.04 m s^{-1} above this depth (Table 7). After 12 hours, most of the uranin was found between 10 and 23 m depth (Figure 10); a small amount was found below this layer.

Since the parameter to which the model is most sensitive, namely, the initial bubble radius r_0 , is not very well known,

TABLE 7. Uranin Experiment and Model Calculation, August 1983

	Model*	Field Experiment
Depth of maximum dye rise (above plume), m		5
Depth of maximum plume rise, m	13–11	
Depth of tracer insertion, m		10–23
Equilibrium depth, † m	30–25	
Plume velocity, m/s		
57–40 m depth	0.12–0.13	0.16
40–14 m depth	0.06–0.08	
40–5 m depth		~0.04

*Parameters: initial bubble radius, 2–4 mm; initial plume velocity, 0.15 m/s ($Fr^\circ = 1.6$); oxygen flow, $0.0083 \text{ N m}^3 \text{ s}^{-1}$; depth of diffuser, 57 m; initial cross-sectional area, 20 m^2 ; initial plume radius, 2.52 m; ambient profiles, July 1983 (Figure 6).

†Depth to which the plume water sinks back once it has lost all momentum. The model value was calculated by assuming that no entrainment of ambient water into the plume occurs once the plume has reached its highest level. Therefore, the value given represents a maximum value.

a rigorous test of the model is not possible. However, model results are found to agree reasonably well with observation if r° is taken to lie within the range 2–4 mm (Table 7). Based on visual inspection of the bubbles leaving the diffuser tubes and on the fact that $\approx 1\%$ of the oxygen escaped to the surface, the actual size of r° was estimated to lie within this range. In the lower part of the plume, the observed upward velocity of the dye was close to the modeled plume velocity. In the upper part, the upward velocity of the dye was lower than the modeled plume velocity, and the dye was found to rise to a greater height than that predicted by the model. We presume that small secondary plumes generated by undissolved bubbles remaining after the main plume has stopped rising carried part of the dye up to a depth of 5 m, resulting in a relatively low apparent upward velocity (Table 7). The fact that the tracer was later detected in the lake mainly between the modeled DMPR (11–13 m) and the modeled equilibrium depth (30–25 m) supports this interpretation.

Plume water appears to have mixed with the surrounding water at different rates during the sinking phase. Detailed tests of the model are planned for newer diffuser systems.

4. DISCUSSION

In this section the model is discussed and compared with the results of other investigations in order to explore the range of its validity and to identify open questions.

1. The scientific literature contains just a few experimental data sets which are suitable for comparing model calculations with measurements. The hydrodynamic part of the model was tested by comparing model results with data from the Bugg Spring experiment of *Milgram* [1983]. In this experiment air was injected at 50 m depth through a nozzle of comparatively small area (about 10^{-2} m^2) into a nonstratified system. The model was run with entrainment factors $\alpha_{th} = 0.116$ determined from the observed value α_g and adapted to the top hat distribution by putting $\alpha_{th} = \sqrt{2}\alpha_g$. Calculated and observed values for volume flux, plume radius and plume velocity agreed within 10% everywhere within the vertical profile. The model calculations are also close to the proportionality between plume water volume flux μ and the square root of the airflow rate observed by *Leitch and Baines* [1989]. Since bubble dissolution was not important in their experiment, the bubble solution dynamics was tested by comparison with the experiments of *Motarjemi and Jameson* [1978] in which bubbles of different sizes were released at 10 m depth. Calculated and observed dissolution rates were equal to within a few percent.

2. In this model we do not distinguish between the zone of flow establishment and the zone of the fully developed plume. We also neglect the initial adjustment of the bubble slip velocity just above the diffuser. We try to reduce this insufficiency of the model by choosing appropriate initial plume velocities. There are physical arguments for reducing the range of the initial Froude number Fr° . As displayed in Figure 11 for the case of an unstratified water column, the Froude number (equation (23)) approaches a maximum value of $Fr \approx 1.7$ at about 15–25 m above the diffuser for all initial Froude numbers less than 1.7. In fact, any initial value of Fr°

Fig. 10. Distribution of uranin in Baldeggersee 12 hours after injection into the bubble plume just above the diffuser at 57 m depth. The two diagrams show tracer concentration along two mutually perpendicular transects. Most of the uranin is dispersed between 10 m and 23 m depth. (Courtesy of Naturqua, Bern, Switzerland.)

Fig. 11. Plume velocity and radius, volume flux and local Froude number Fr as functions of depth in unstratified water calculated with standard parameters for artificial mixing (Table 6) for various initial Froude numbers Fr^0 (Fr^0 labeled on curves). Note the contraction and acceleration of the plume for $Fr^0 < 1.0$ and the asymptotic behavior of Fr for $Fr^0 < 1.7$.

> 1.7 is unrealistic because the plume would have to undergo a decrease in velocity immediately above the diffuser (just where bubbles enter the flow), as can be seen in Figure 11a for the case $Fr^0 = 2.5$. For $Fr^0 < 1.5$ the plume is accelerated immediately above the diffuser, as demonstrated in Figure 11a for the case $Fr^0 = 1.0$. For $Fr^0 \approx 1.0$ the plume initially undergoes contraction as it rises (Figure 11b). This contraction becomes more pronounced as Fr^0 decreases to 0.5. Because a slight contraction can be observed in tank experiments [Müller *et al.*, 1987], we regard values of 0.75–1.0 as a reasonable lower bound for Fr^0 .

The initial Froude number Fr^0 depends most likely on the geometry of the diffuser source, since for the zone of flow establishment it is relevant whether the water can flow upward through the diffuser area (open source, diffuser above the bed) or whether the water has to flow horizontally to the rising plume (closed source, diffuser resting on the bed). It seems likely that in the latter case the initial Froude number Fr^0 will be smaller. Because the slight contraction which was observed in tank experiments occurred for a closed source [Müller *et al.*, 1987] and because the lake restoration diffuser unit with widely spaced tubes represents an open source, we conclude that the initial Froude number for the latter lies between 1.0 and 1.7.

3. We assumed top hat distributions for all plume properties. Mathematically, the scaling involved here is nearly equivalent to that inherent in the horizontally integrated model, presented by Ditmars and Cederwall [1974]. It simplifies the model, but does not restrict its dynamic generality, as long as the profiles (of velocity and buoyancy) remain similar at all heights. We introduced λ , the ratio of the inner plume radius (containing the bubbles) to the velocity radius, as the only plume geometry parameter. However, as long as

the water velocities in the bubble core and the outer annulus are equal, the model is independent of λ , as is evident from (16). We retained λ in this model, however, in order to obtain the correct equations for O_2 and N_2 if the double plume model [McDougall, 1978] is used.

4. In the case of strong stratification, single-phase plume models do not adequately describe the plume if the bubbles are not completely dissolved at the predicted depth of maximum plume rise. Bubble plume experiments carried out by McDougall [1978] in a stratified tank showed that fluid from the outer annulus can leave the plume at levels of neutral buoyancy and spread out horizontally, while the center of the plume, where all the bubbles are concentrated, continues to rise. The present model disregards the double plume structure, the detrainment, and the development of secondary plumes initiated by the remaining bubbles. Secondary plumes are created when oxygenation is carried out with relatively large bubbles which are not yet dissolved at DMPR, or in the case of artificial mixing, when the airflow rate is not large enough to bring the entire plume to the surface. For such situations, the model presented here loses its validity as soon as the water has lost its vertical momentum. However, oxygen input with adequately small bubbles which disappear completely (Figures 7c and 7e) will not generate secondary plumes. The price for using the double plume model is the addition of a further two entrainment factors.

5. The entrainment coefficient α is not a constant for all plumes. A simple parameterization is not obvious. In our model for the standard case $\alpha_{th} = \sqrt{2}\alpha_g = 0.11$ was used, corresponding to $\alpha_g = 0.08$ as given by Fischer *et al.* [1979]. Milgram [1983] calculated α for bubble plumes of various scales based on data from several authors and correlated

these values with a "bubble Froude number F_B " defined by local plume properties. Milgram's [1983] values for α_{th} averaged 0.081 and 0.12 for airflow rates of 0.024 and 0.118 $N\ m^3\ s^{-1}$, respectively, for the 50-m-deep Bugg Spring experiment. Fannelop and Sjoen [1980] observed values of α_{th} from 0.10 to 0.14, increasing with airflow rates in the range 0.005–0.022 $N\ m^3\ s^{-1}$.

6. In the conservation equations used here, vertical turbulent flux terms (such as $w'c'$) are neglected compared to fluxes based on mean properties, presuming that the former are small compared to the latter. This assumption is equivalent to a momentum amplification factor of $\gamma = 1$ [Milgram, 1983]. The analysis of Milgram [1983, Table 1] yielded values of $\gamma = 1.0$ –1.1 for typical oxygenation gas rates and $\gamma = 1.1$ –1.26 for gas rates appropriate to artificial mixing. The corresponding overestimation of the momentum gain would consequently be not more than 21%. However, Milgram [1983] has shown that for plumes with low bubble density, γ can increase dramatically.

7. There are many questions concerning bubble plumes still remaining open which will have to be addressed in tank and field experiments. Apart from the problems raised in the discussion above, we would like to mention the question of how bubble size affects bubble dynamics [Müller et al., 1987], the problem of continuous detrainment and problems related to the interaction of the plume with the ambient flow field in the lake (cross flow, plume wandering). Some of these questions are being investigated in the new bubble-plume tank of the Swiss Federal Institute of Technology [Müller et al., 1987]. We think that the mathematical model presented here provides a useful framework for further investigations and helps to relate laboratory experiments to field situations when bubble dissolution and expansion are important.

5. SUMMARY AND CONCLUSIONS

1. A steady, horizontally averaged plume model was developed based on the entrainment hypothesis, variable buoyancy determined by ambient profiles and actual plume properties, and processes of bubble gas exchange and decompression. The model was applied to a stratified lake for a diffuser system with a large open source and low gas flow rates (of the order of $5 \times 10^{-4}\ N\ m^3\ s^{-1}\ m^{-2}$). Model calculations were in encouraging agreement with experimental data.

2. Gas exchange, decompression and bubble slip velocity affect both bubble volume and bubble residence time. Consequently, buoyancy flux is a complex function of height. For small bubbles, gas exchange determines plume dynamics to a large extent.

3. Apart from the initial bubble size, the gas flow rate and entrainment coefficient are the parameters to which the model is most sensitive. The sensitivity to diffuser size and initial velocity turned out to be small, provided the initial velocity was chosen below an upper bound. We argue that a reasonable range of the initial velocity is given by the condition $1 < Fr^\circ < 1.7$ for an open source (diffuser situated above the bottom), and $Fr^\circ < 1$ for a closed source (diffuser resting on the bottom).

4. Model calculations clearly define two modes of diffuser operation: oxygenation (or aeration) and artificial mixing. Optimal bubble size depends crucially on the objectives

and on the radius-dependent parameters mass transfer and slip velocity. In order to prevent oxygen loss by escaping bubbles and to achieve insertion into the deepest layers for oxygenation, bubbles of radius $\approx 1\ mm$ are the best choice (here about 0.8 mm). In contrast, artificial mixing is most efficient for an initial bubble radius of about 1 cm.

5. Given the operational goals, the model helps to optimize system design in order to conduct the internal restoration of a lake as economically as possible.

6. The model can be used as a tool to design lake aeration systems and evaluate sensitivity to operation parameters. It can also help in transferring laboratory results to the field. Finally, it serves to define areas where additional research is needed.

Acknowledgments. We are grateful to A. Müller, A. Gyr and C. Hugli for stimulating discussions, Naturaqua Bern for providing Figure 10 and D. Livingstone for improving the English. This work was supported by Swiss National Science Foundation grant 20-27751.89.

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(Received December 4, 1991;
revised July 6, 1992;
accepted July 15, 1992.)