Multifractality in spin glasses

We unveil the multifractal behavior of Ising spin glasses in their low-temperature phase. Using the Janus II custom-built supercomputer, the spin-glass correlation function is studied locally. Dramatic fluctuations are found when pairs of sites at the same distance are compared. The scaling of these fluctuations, as the spin-glass coherence length grows with time, is characterized through the computation of the singularity spectrum and its corresponding Legendre transform. A comparatively small number of site pairs controls the average correlation that governs the response to a magnetic field. We explain how this scenario of dramatic fluctuations (at length scales smaller than the coherence length) can be reconciled with the smooth, self-averaging behavior that has long been considered to describe spin-glass dynamics.

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Significance

Many seemingly irregular objects (coast shores, for instance) look the same at different observation scales. In many cases, a single number, the fractal dimension, characterizes the scale changes. Other systems, known as multifractals, need a continuous range of parameters to characterize the change of scale. Multifractal behavior has been identified in a plethora of situations, from human heartbeats to financial time series, and is often accompanied by large statistical fluctuations. Spin glasses are one of the best-studied model systems for complexity, and large statistical fluctuations are completely absent from their dynamics. Our finding of multifractal scaling in the spin-glass off-equilibrium dynamics is, therefore, surprising. The paradox is solved through the concept of coherence length.

The authors declare no competing interest.

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Fig. 1. The correlation function, Eq. 10, as computed for the three-dimensional Ising diluted ferromagnet (DIL) and for the Ising spin glass (EA), versus distance $r$. Data were obtained in systems of linear size $L = 160$ with coherence length $\xi(t_w) = 20$ (dashed vertical line) at temperature $T = 0.9$—recall that $T_c \approx 1.1$ for EA (32). As explained in Methods, the coherence length is computed from the integral $\xi = \int_0^\infty r^2 C^{(g)}(r) \, dr$ (the integrand is shown in the SI Appendix).

Fig. 2. Correlation function $C^{(g)}(r = \xi(t_w))$, see Eq. 10 versus the coherence length $\xi(t_w)$, as computed for DIL (Top) and for EA (Bottom) at temperatures $T = 0.9, 0.8$ and 0.7 (see Methods for a complete definition of $T$). Error bars are smaller than the point size. The dashed line is our fit to Eq. 2, with $q = 1$, for EA at $T = 0.9$ (to avoid scaling corrections, we fit in the range $\xi(t_w) \in [10, 20]$, see SI Appendix for further information). Note that, while the DIL $C^{(g)}(r = \xi(t_w))$ tends to a $T$-dependent positive limit for large coherence length (which excludes multiscaling at $T < T_c$), the spin-glass correlation functions steadily decrease with $\xi(t_w)$.

Fig. 3. Ratio of the second moment of the spin-glass correlation function $C_4$ computed at $r = \xi(t_w)$, $C_4^2$, to the squared first moment, $C_4^2$, as a function of $C_4$. We show the data for all temperatures considered in this work. $C_4$ tends to zero as the coherence length $\xi(t_w)$ gets large, recall Fig. 2. Note that $C_4^2/C_4$ scales with $C_4$ as a power law, which indicates that in the scaling limit (i.e., $\xi(t_w) \to \infty$ or $C_4 \to 0$) the order of magnitude of $C_4^2$ is larger than the one of $C_4^2$. Data in the glassy phase, $T < T_c$, roughly follow the same scaling curve. At the critical point, there is still a power type relation with a slightly different exponent. Error bars are smaller than the points size. The same data are shown as a function of $\xi(t_w)$ in SI Appendix.
of this modulating factor. If continued to \( C_4 \rightarrow 0 \) (i.e., as \( \xi(t_w) \) grows, see Fig. 2), this power law implies that the orders of magnitude of \( C_4^2 \) and \( C_4^2 \) differ in the large-\( \xi(t_w) \) scaling limit. This behavior is not reminiscent of a monofractal, which in the scaling limit is characterized by a single quantity (say, \( C_4^0 \)).

We also note from Fig. 3 that all our data with \( T < T_c \) follow the same scaling curve, which slightly differs from its counterpart at the critical point. This is not completely unexpected, because the \( \epsilon \)-expansion tells us that the average \( C_4 \) at \( T_c \) decays as a power law with distance with an exponent (40) that is twice as large as the exponent for \( T < T_c \) (41). In fact, we lack an explanation for the similarity of the two exponents that can be observed in Fig. 3. From now on, our analysis will focus on our data at \( T = 0.9 \), namely the temperature in the spin-glass phase where we are able to reach the largest \( \xi(t_w) \).

A picture of the physical situation is presented in Fig. 4. We may expect a different behavior for the average and the local correlation function when distances up to \( r \sim \xi(t_w) \) are considered \( [\theta(T = 0.9) \approx 0.4(27)]^{\#} \)

\[
C_{4}^{w}(r, t_w) \sim \frac{1}{r^\beta}, \quad C_{4}(x, x + r; t_w) \sim \frac{1}{r^{\beta}M(x, x; t_w)}. \quad [1]
\]

As the reader can check from Fig. 4, the order-of-magnitude modulating factor \( M(x, r, t_w) \) varies by a factor of 16, which indicates that there are site pairs \((x, x + r)\) a lot more—or a lot less—correlated than the average. In fact, see Fig. 5, the median correlation function at distance \( r = \xi(t_w) \), scales as \([M_{4}]^\alpha \), with \( \alpha \approx 1.5 \). In other words, the typical correlation function is a lot smaller than the average value.

\[\frac{\text{Grayscale representation of the order-of-magnitude modulating factor } M(x, r, t_w), \text{ see Eq. } 1, \text{ computed for site } x = (64, 64, 64) \text{ of a sample with coherence length } \xi(t_w) = 20, \text{ at } T = 0.9, \text{ with an } N_0 = 512 \text{ estimator (Methods). We show results for displacement vectors } r = (r_x, r_y, r_z) \text{ in a cube } -40 \leq r_x, r_y, r_z \leq 40. \text{ The Top-Left panel depicts the three visible faces of the cube, while the other three panels show sections at } r_z = -20, 0, 20, \text{ respectively. Our color code is darker the smaller } M(x, r, t_w) \text{ (hence, the more slowly correlations decay with distance). For ease of representation, we have chosen a color code linear between the minimal value of } M(x, r, t_w) \text{ and } 2.5. \text{ Displacements } r \text{ with } M(x, r, t_w) > 2.5 \text{ are depicted as if } M(x, r, t_w) = 2.5. \text{ See SI Appendix for more examples of this modulating factor.}\]

\[\text{\#The correlation function behaves as } C_{4}^{w}(r, t_w) = G(r; \xi(t_w)) \cdot r^\beta \text{ for large } r, \text{ where the cut-off function } G(r) \text{ decays faster than exponentially as } x \text{ grows [see, e.g., refs. 42 and 43]. Hence, for } r \sim \xi(t_w) \text{ one may consider either power-law scaling in } r \text{—as in Eq. 1—or in } \xi(t_w)\text{—as in Eq. 2. The analysis of scale invariance in a fractal (or multifractal) geometry typically involves power laws.}\]
In order to make the above qualitative description quantitative, we consider the moments of the probability distribution $P_{\xi}$ at distance $r = \xi(t_w)$. The $q$-th moment turns out to follow a scaling law

$$C_q \sim A_q \xi^{-q}.$$

Eq. 2 defines the large-deviations function $f(a)$. Then, we find for the moments of $C_q$

$$C_q = \int_0^1 d\xi P(\xi)C_q^\xi \sim \int_0^\infty da \frac{\xi^q}{f(a)-qa}.$$

For large $\xi$, the above integral is dominated by the maximum of $f(a)$ at some value $a = a^*:

$$C_q \sim \frac{1}{\xi^{f(a^*)+qa^*}}.$$

Comparing with Eq. 2, we realize that $f(a)$ is just (minus) the Legendre transform of the singularity spectrum $\tau(q)$:

$$f(a) = -\max_q \{\tau(q) - qa\}.$$

We show $f(a)$ in Fig. 7, as computed from our fitting ansätze $\tau_1(q)$ and $\tau_2(q)$, in Eq. 3. In the range of Fig. 6—since $\alpha(q) = \tau(q)$—the results from the two ansätze can hardly be distinguished. The two, however, differ in that the range of $\alpha$ for $\tau_2(q)$ goes all the way down to $\alpha = 0$ (because $\alpha_2(q) = d\tau_2/da \sim 1/q$). Indeed, if $\tau(q)$ goes as $log(q)$ for large $q$, then the large-deviations function goes as $f(0) \sim log(a)$.

Let us recapitulate: the probability of finding a site $x$ with $C_q(x,x+r)$ scaling as $1/r^{2q}$ for $r \sim \xi$ goes in the scaling limit as $4^{-q}$. There are, hence, a lot more sites displaying the median scaling exponent $\alpha \approx 0.65$ than there are for the average scaling $\alpha \approx 0.4$ (because $f(0.65) > f(0.4)$, recall Fig. 5). The larger $\xi(t_w)$ grows, the more pronounced this difference is. Thus, the expression “silent majority” (44) could be aptly employed to at fitting our data (see again SI Appendix), only $\tau_2(q)$ displays the logarithmic growth with $q$, at large $q$, that we find more plausible.

**Discussion**

Following ref. 9, we shall discuss our results in terms of a different stochastic variable, $a = log C_q(r = \xi(t_w))/log[1/\xi(t_w)]$, so that (we drop the argument in $\xi$ for the sake of shortness)

$$C_q = \frac{1}{\xi^a}, \quad P(C_q)\frac{dC_q}{da} \sim \xi^\tau(a).$$

Eq. 4 defines the large-deviations function $f(a)$. Then, we find for the moments of $C_q$

$$C_q = \int_0^1 d\xi P(\xi)C_q^\xi \sim \int_0^\infty da \frac{\xi^q}{f(a)-qa}.$$

For large $\xi$, the above integral is dominated by the maximum of $f(a) - qa$ at some value $a = a^*$:

$$C_q \sim \frac{1}{\xi^{f(a^*)+qa^*}}.$$

Comparing with Eq. 2, we realize that $f(a)$ is just (minus) the Legendre transform of the singularity spectrum $\tau(q)$:

$$f(a) = -\max_q \{\tau(q) - qa\}.$$
describe spin-glass dynamics: the central limit theorem ensures that it is the (somewhat exceptional) average value the one that can be measured on length scales larger than \( \xi(t_w) \) (hence, in experiments). The experimental-scale dynamics is, however, not completely blind to these short-scale fluctuations. Indeed, temperature chaos (30)—and, hence, rejuvenation (28), which is certainly experimentally observable (see, e.g., ref. 29)—is ruled by statistical fluctuations at the scale of \( r \) smaller than, or similar to, \( \xi(t_w) \).

Our data show that varying \( T \) simply changes \( \tau(q) \) by an essentially constant factor [e.g., \( \tau(q, T_c) \approx 1.5 \tau(q, T = 0.9) \), see SI Appendix]. Furthermore, Fig. 3 makes us confident that, taking \( C_4 \) as scaling variable instead of \( \xi(t_w) \), the overall picture is essentially temperature independent for \( T < T_c \).

Whether or not multifractal behavior is also present in equilibrium correlation functions in the spin-glass phase stands in essentially open question. Statics-dynamics equivalence (26, 45–47) suggests that the answer will be positive.

As a final remark, let us stress that ongoing efforts to build a mathematically rigorous theory of nonequilibrium spin-glass dynamics through the concept of the maturation metastate (see ref. 48 and references therein) should take into account the extreme spatial heterogeneity unveiled in this work.

Materials and Methods

Model and Simulations. We focus on the Edwards-Anderson model (EA) in a simple cubic lattice with linear size \( L = 160 \) and periodic boundary conditions. Our \( S_x = \pm 1 \) spin, placed at the lattice sites, interact with their nearest neighbors through the Hamiltonian:

\[
\mathcal{H} = - \sum_{\langle x,y \rangle} J_{xy} S_x S_y .
\]  

The coupling constants \( J_{xy} \) are independent random variables \( J_{xy} = \pm 1 \) with equal probability), fixed once and for all at the beginning of the simulation (this average over samples is denoted by an overline \( \langle \cdots \rangle \)).

We shall use \( 16 \) samples in this work. In general, errors will be computed with a jackknife method over the samples (see, for instance, refs. 49 and 50). We use the shorthand \( C_4^S(t_w) \) to indicate this average over the three equivalent displacements \( r = (r, 0, 0) \) and permutations. We shall use the shorthand \( C_4^r(t_w) \) to indicate this average over the three equivalent displacements \( r \).

To compute the coherence length \( \xi(t_w) \), we follow \( 27, 42, 50 \) and compute the integrals

\[
l_n(t_w) = \int_0^\infty \rho^2 C_4^r(r, t_w) d\rho .
\]  

Then, \( \xi(t_w) = l_2(t_w)/l_1(t_w) \).

As stated above, we have simulated, as a null experiment, the link-diluted Ising model (DIL). The only difference with the Hamiltonian in Eq. 8 is the choice of the couplings: \( J_{xy} = 1 \) (with 70% probability) or \( J_{xy} = 0 \) (with 30% probability). Since all couplings are positive or zero, this is a ferromagnetic system without frustration. All our simulation and analysis procedures are identical for the DIL and EA models. The critical temperature is \( T_{DIL} = 3.0609(5) \) (52).

Table 1. Maximum \( t_w \) and coherence length reached for each of our models and simulation temperatures

<table>
<thead>
<tr>
<th>( T ) or ( \tilde{T} )</th>
<th>( t_w ) (EA)</th>
<th>( \xi_{\text{max}} ) (EA)</th>
<th>( t_w ) (DIL)</th>
<th>( \xi_{\text{max}} ) (DIL)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.7</td>
<td>46,531,866,276</td>
<td>12</td>
<td>498</td>
<td>15</td>
</tr>
<tr>
<td>0.8</td>
<td>18,734,780,191</td>
<td>15</td>
<td>919</td>
<td>21</td>
</tr>
<tr>
<td>0.9</td>
<td>15,172,184,825</td>
<td>20</td>
<td>954</td>
<td>23</td>
</tr>
</tbody>
</table>

Cubic symmetry, present in averages over the samples, allows us to average over the three equivalent displacements \( r = (r, 0, 0) \) and permutations. We shall use the shorthand \( C_4^r(t_w) \) to indicate this average over the three equivalent displacements \( r \).

Unbiased Estimators of Powers of \( C_4(x, y, t_w) \). Given \( x \) and \( y \), we need an unbiased estimator of \( C_4^r(x, y, t_w) = \langle S_x(t_w)S_y(t_w) \rangle^2 \).

Table 1. Maximum \( t_w \) and coherence length reached for each of our models and simulation temperatures

\[
\text{Table 1. Maximum } t_w \text{ and coherence length reached for each of our models and simulation temperatures}
\]
\( \{ C_q(x, y, t_w) \}_q \) is an unbiased estimator of \( \{ S_k(t_0) S_{k+r}(t_0) \}_r^2 \) because it is an average over all possible (poor, but unbiased) estimators in Eq. 12. Our computation of \( P(N_B; M; S = 2q, p) \) is explained in the SI Appendix.

The Probability Distribution of the Correlation Function. We wish to study the probability distribution function (pdf) for \( \{ S_k(t_0) S_{k+r}(t_0) \}_r^2 \) (periodic boundary conditions are assumed for \( x + r \)). We have only considered displacements \( r = (r, 0, 0) \) and permutations - and have chosen the measuring times in such a way that \( r = (x, t_0) \).

Note that, given the starting point \( x \) and the sample \( \{ S_k(x) \}_k \), \( \{ S_k(x) S_{k+r}(x) \}_r^2 \) is not a fluctuating quantity. Hence, we are referring to the pdf as \( \{ P(M; N_B, x) \}_x \), namely the probability, as computed over the starting point \( x \) and the samples, that exactly \( M \) of the \( N_B \) signs \( \{ S_k(x) \}_k S_{k+r}(x) \) turn out to be \(-1\) in our simulation of this specific sample. Hence, the unbiased estimator of the \( q \)th moment of \( \{ S_k(t_0) S_{k+r}(t_0) \}_r^2 \) with \( r = (x, t_0) \) is

\[
C_q^2(x, t_0) = \frac{N_B}{N_0} - \sum_{M=0}^{N_0} P(M; N_B, x)(G(N_B, M, q),
\]

where \( G(N_B, M, q) \) was defined in Eq. 14.

Unfortunately, the median of the pdf for \( \{ S_k(t_0) S_{k+r}(t_0) \}_r^2 \) is more difficult to compute. Our strategy, explained in full detail in SI Appendix, consists in computing biased estimators of the median, with bias of order \( 1/N_B \). Then we compute these biased estimators for a sequence \( N_B = 32, 64, 128, 256, \) and 512, and proceed to an extrapolation \( N_B \to \infty \). We obtain the \( \{ P(M; N_B, x) \}_x \) from \( N_B = 512 \) counterpart as

\[
P(M; N_B, x, \xi) = \frac{N_B}{N_0} - \sum_{M=0}^{N_0} P(M; N_B, x, \xi) P(N_B, M; S = N_B, p = M).
\]

The probabilities \( P(N_B; M; S, p) \) were defined in the previous subsection in this Methods section.

Computation of \( r(q) \). In order to minimize convolutions to scaling, we have fitted the normalized moments as

\[
\log C_q^2(x, y, t_w) / \log C_q^2(y) = 2 \left( \frac{a (x, y)}{\sqrt{a^2 + b^2 + c^2}} \right),
\]

where \( C_q^2(x, y, t_0) \) is interpolated to noninteger arguments using a fit obtained from data with integer \( q \).

Data, Materials, and Software Availability. All study data are included in the article and/or SI Appendix. The data and the scripts that generate the figures in the main text can be downloaded from https://github.com/JanusCollaboration/multifractal_EA/tree/main/MAIN (53). The data and the scripts that generate the figures in the SI Appendix can be downloaded from https://github.com/JanusCollaboration/multifractal_EA/tree/main/SI (54).

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