Fatigue crack arrest in steel beams using FRP composites

Seyed Mahdi Hosseini\textsuperscript{a}, Jakob Melchior\textsuperscript{b}, Mohammadreza Izadi\textsuperscript{c,*,}\textsuperscript{a}, Elyas Ghafoori\textsuperscript{d,e,*}

\textsuperscript{a} Schulich School of Engineering, University of Calgary, Calgary, Canada
\textsuperscript{b} Concretum Construction Science AG, Zürich, Switzerland
\textsuperscript{c} School of Civil & Environmental Engineering, University of New South Wales, UNSW, Sydney, Australia
\textsuperscript{d} Structural Engineering Research Laboratory, Swiss Federal Laboratories for Materials Science and Technology (Empa), Duebendorf, Switzerland
\textsuperscript{e} Institute of Structural Engineering (IBK), Department of Civil, Environmental and Geomatic Engineering, ETH-Zürich, Zürich, Switzerland

ARTICLE INFO

Keywords:
Carbon-fiber reinforced polymer (CFRP)
Pre-stressing
Fatigue crack
Fracture
Metallic structures

ABSTRACT

Previous studies have demonstrated the effectiveness of strengthening with pre-stressed carbon-fiber reinforced polymer (CFRP) composites to increase the lifetime of cracked steel members. In some cases, complete crack arrest has been observed. This study aims to present a method that can estimate the minimum required prestressing that would result in a complete crack arrest in steel I-beams. Analytical and numerical models based on linear elastic fracture mechanics (LEFM) were developed and verified using a set of experimental results. Three steel I-beams with different crack lengths were strengthened with pre-stressed CFRP composites and later tested under a high-cycle fatigue loading regime. It was shown that the pre-stressed CFRP composites could result in a crack closure mechanism, in which the crack surfaces remained closed even under large external loads. Furthermore, it was shown that by considering the stiffness of the CFRP in the analytical formulation, the amount of prestressing required to arrest the fatigue crack growth can be reduced.

1. Introduction

Aging is becoming a worldwide concern for existing steel bridges. Approximately 22% of the bridges in Europe are made of steel, and thus they are prone to aging-related problems such as fatigue cracking and require constant inspections and repairs [1]. Traffic loads have increased over the last few decades, and this is especially critical for bridges that are already older than 50 years as fatigue damage is accumulated over time. In some cases, the damage is so advanced that the entire structure needs to be replaced. This often requires a huge financial investment and may also lead to traffic congestion; therefore, a repair seems to be a profitable alternative whenever possible [2–4].

A very recent repair option, favored by bridge engineers, is the use of carbon-fiber reinforced polymer (CFRP) laminates [5–8]. The advantages lie in the high strength-to-weight ratio of the composite material along with its good fatigue behavior. Among the CFRP-strengthening techniques (with bonded and un-bonded anchorages), the most efficient solution has been achieved by pre-stressing the CFRP composites [7,9,10]. This not only reduces the stress concentrations in the problematic zone but also compresses the entire area,
Nomenclature

\( a \)  
\( a_c \)  
\( a_f \)  
\( b \)  
\( b_f \)  
\( c \)  
\( e \)  
\( h \)  
\( h_0 \)  
\( l_1, l_2, l_3 \)  
\( m \)  
\( p_i \)  
\( r_p \)  
\( t_f \)  
\( t_w \)  
\( A_c \)  
\( A_{CFRP} \)  
\( A_{St} \)  
\( A_{1,St}, A_{2,St} \)  
\( B, B' \)  
\( C, C' \)  
\( E \)  
\( E_{CFRP} \)  
\( E_{St} \)  
\( F \)  
\( F_{Ca} \)  
\( F_{max} \)  
\( G^* \)  
\( H \)  
\( I_{Z,St} \)  
\( J_S \)  
\( K_i \)  
\( K_{op} \)  
\( K_{th} \)  
\( K_{\max}, K_{\min} \)  
\( K_{op} \)  
\( L \)  
\( L_i \)  
\( L_f \)  
\( M_i \)  
\( M_{\text{max}} \)  
\( N \)  
\( P_{CFRP}^u \)  
\( R \)  
\( T \)  
\( U_T \)  
\( Y_c \)  
\( \varepsilon \)  
\( \theta \)  
\( \theta' \)  
\( \phi \)  
\( \chi_1, \chi_2, \chi_3 \)  
\( \lambda_1, \lambda_2, \eta_1, \eta_1 \)  
\( \nu \)  
\( \sigma \)  
\( \sigma^*_{CFRP} \)  
\( \varphi \)  
\( X_1, X_2, X_3 \)  
\( \Delta K \)  
\( \Delta K_{eff} \)
resulting in an active closure of already existing fatigue cracks [11]. A closed-form analytical solution for the flexural and interfacial behavior of metallic beams strengthened with pre-stressed CFRP plates was developed by Ghafoori and Motavalli [12] and verified using experimental tests. It was also shown that the intermediate adhesive between the CFRP and the steel substrate is the weak zone of the bonded retrofit system [13,14]; thereby, the un-bonded anchorages were privileged to be an alternative for anchoring the CFRP laminates.

The effect of this retrofitting method on the fatigue performance of CFRP-strengthened bridge steel members was demonstrated by Tavakkolizadeh and Saadatmanesh [15] for notched beams under four-point bending subjected to a range of frequencies and stresses. The results showed a more than threefold extension of the fatigue life, together with a reduction in the crack propagation rate. Similarly, Huawen et al. [16] demonstrated a fourfold increase in the fatigue life of tension steel plates strengthened with pre-stressed CFRP. Furthermore, Täljsten et al. [17] investigated the strengthening of aged structures with non-prestressed and pre-stressed CFRP. For the experimental study conducted by Täljsten et al., the specimens were cut from an old steel girder and strengthened with both pre-stressed and non-prestressed CFRP. Additionally, several studies were carried out to develop an analytical solution to predict the fatigue crack growth and fatigue life of the CFRP strengthened steel plates [18,19].

While the non-prestressed specimen showed a significant increase in fatigue life, strengthening with pre-stressed CFRP resulted in a complete halt of crack propagation. Simple fracture mechanics formulations were used to predict this behavior by superimposing multiple stress intensity factors (SIFs). Ghafoori and Motavalli [20] developed an analytical model for the SIF of cracked I-profile beams based on the crack surface widening energy release rate. The analytical model can be used to evaluate the effect of strengthening on the stress state of the crack. The analytical model was verified using experimental results.

A limited number of studies have already looked at the fracture and fatigue performance of structures strengthened with un-bonded pre-stressed CFRP [21]. These studies have looked at the crack initiation and crack propagation in strengthened structures, but there is no systematic study, to the best of our knowledge, on the possibility of arresting already existing fatigue cracks using this method. This study, which is based on [22], aims to investigate the behavior of fatigue cracks with external strengthening using pre-stressed CFRP. The factors influencing the crack length and the magnitude of the pre-stressing were considered in the study program. To achieve these goals, laboratory tests were planned and executed. In addition to the experiments, a numerical analysis was performed to verify the experimental results and to study the aspects that were not included in the experimental tests. The work also involved expanding a previously developed analytical model and providing a formulation for calculating the minimal pre-stressing level that is needed to arrest a fatigue crack. Unlike its previous analytical counterparts, in particular, the works of Ghafoori et al. [23], the new formulation takes into account the stiffness of the CFRP. This assumption leads to the redistribution of forces based on the amount of external forces acting on the beam. Therefore, the external force acts in the strengthening by increasing the prestressing force in the CFRP and reducing the SIF. As a result, the amount of required pre-stressing to arrest a crack determined in this case is lower than the value calculated in the aforementioned references, which differentiates this study from all the previous works in this field.

2. Experimental plan

2.1. Four-point bending test setup

The static and cyclic tests on all the beams were performed using the same four-point bending testing setup (Fig. 1). For the support on the sides, rocker bearings were used, and the free span of the beam between the supports was 1,200 mm. These supports allowed the beam to rotate around the out-of-plane axis at both ends while restricting displacements. For the force application, two hydraulic cylinders, each placed 250 mm inside the supports, were used. To guarantee good accuracy, cylinders with static capacities of 20 kN and dynamic capacity of 10 kN were used for low load levels, and cylinders with capacities of 50 and 100 kN for static and fatigue loading, respectively, were used when higher load levels were required.

2.2. Instrumentation and measurements

Different types of strain gauges were used to measure the strain at different locations on the specimens. One electrical strain gauge of type 1-LY66-6/120 (produced by Hottinger Baldwin Messtechnik GmbH) was glued to the center of the CFRP. The type 1-LY66-6/120 strain gauges have a k-factor of 2.02 ± 1.0% and a resistance of 120Ω ± 0.3%. For the strain measurements at the upper flange of the beam, a type FGMH-1 (Tokyo Sokki Kenkyujo Co. Ltd) gauge was used. The strain gauge was placed in the center of the upper
flange (lengthwise and across). The FGMH-1 strain gauges have a k-factor of 2.02 ± 2.0% and a resistance of 120 Ω ± 0.5%. To measure the strains around the crack, smaller type 1-LY11-0.6/120 strain gauges (produced by HBM GmbH) were used as they could be placed closer to the crack. The type 1-LY11-0.6/120 strain gauges have a k-factor of 1.74 ± 1.0% and a resistance of 120 Ω ± 1.0%. Fig. 1 illustrates all the strain measurements on the specimen with short (<10 mm) and long (>10 mm) cracks.

A vertical extensometer (WSF 50 mm by HBM Messtechnik) was used to measure the vertical deflection at the mid-span of the beam. The extensometer has a measuring range of ±25 mm. To measure the crack opening, an extensometer with a tension spring (PE8301 by COMPAC) was used, which was attached to the side of the bottom flange. The extensometer has a measuring range of 2 mm. Optical measurements were also conducted to measure the crack length. A measuring microscope with a build-in 8X magnifying glass was used, which allowed photographing the crack tip with a larger magnification than that of the camera lens alone. To be able to see the crack more clearly, the area around the desired crack was painted in white using a thin layer of common acryl spray paint.

2.3. Materials and anchorage system

The IPE 120 steel profiles were provided by Briner AG, Winterthur, Switzerland. All beams were from the same production batch to reduce any variance in the material. The ordered steel grade was S355, and Table 1 presents the mechanical properties according to EN 10025. The CFRP laminate used for the strengthening in this project was produced by S&P Clever Reinforcement Company, Switzerland, with the product name 150/2000 type 50/1.2S laminate. The laminates were 50 mm wide and had a thickness of 1.2 mm. Table 2 presents the mechanical properties of the material testing sheet provided by the S&P AG Company.

For the attachment between the CFRP and the bottom flange of the beam, custom-made friction clamps were used, which were already available from a previous project at Empa [24]. The clamps were produced by Mettler Mech. Produkte GmbH, S355J0 steel (Fig. 1). The clamps were 80 mm wide and utilized four M20 bolts tightened to a torque of 480 Nm to create the required tensile force. To further increase the friction between the steel substrate and the CFRP, double-sided sandpaper (P180 grid size) was placed between each of the surfaces. EKagrip® friction shims produced by ESK Ceramics GmbH & Co. KG were used. This arrangement results in a theoretical capacity of 303.4 kN for the clamp which is almost 190% of the ultimate tensile strength of the CFRP and, therefore, more than sufficient.

<table>
<thead>
<tr>
<th>Steel grade</th>
<th>Yield strength [MPa]</th>
<th>Tensile strength [MPa]</th>
<th>Young’s modulus [GPa]</th>
</tr>
</thead>
<tbody>
<tr>
<td>S355JR</td>
<td>355</td>
<td>510</td>
<td>210</td>
</tr>
</tbody>
</table>
2.4. Pre-stressed CFRP strengthening technique

To strengthen the beams with the unbonded pre-stressed CFRP, a set-up was prepared that allowed a constant strain to be applied while the CFRP was attached to the bottom flange of the beam using the metal clamps. Therefore, the CFRP was placed in a rigid pre-stressing frame with an actuator placed on one end to supply the required displacement for the strain application (Fig. 2 represents the pre-stressing test setup used for strengthening). End clamps were used to connect the ends of the CFRP to the threaded rods. A single layer of sandpaper (P180 grid size) was placed between the CFRP and clamps to increase the friction, and the clamps were tightened up to 120 N.m using a torque wrench. To prevent the clamps from rotating during the tightening process, which might damage the CFRP, they were fixed to the pre-stressing frame using screw clamps. The pre-stressing load was monitored with a load cell integrated into the setup and the strain gauge on the CFRP. Next, the beam was placed on the top of the CFRP, and the end anchorage clamps were tightened to the beam. Subsequently, the actuator load was relaxed, resulting in the load cell reading going down to zero. During this relaxation, the load from the pre-stressed CFRP was transferred to the bottom flange of the beam, resulting in a slight upward deflection of the beam and reduction of the pre-stressing stress in the CFRP.

2.5. Test procedure

Three different specimens were prepared and tested in this study to cover the two influencing parameters of crack length and pre-stressing level. The specimen layouts are summarized in Table 3.

2.5.1. Pre-cracking

The bottom flange of each beam was cut through at the mid-span, and a 1.5-mm wide and 6-mm long notch was cut into the web. The notch was pointed at the tip to create a starting point for the initial crack to grow owing to the stress concentration. The fillet next to the notch was removed, and the adjacent area sanded down to provide a smooth surface (Fig. 3). The crack in the web of the specimen was grown by subjecting the beams to cyclic loading under the same four-point bending setup used for the later tests.

The pre-cracking was performed based on the ASTM code E647–08 in the “Standard Test Method for Measurement of Fatigue Crack Growth Rates” [37]. The code requires a minimal pre-crack length of 10% of the specimen width, notch width, or 1 mm, whichever is the greatest. The load needs to be applied symmetrically to the notch to guarantee that the crack growth is straight and perpendicular to the flange. It is also suggested that the crack propagation rate be maintained below $10^{-8}$ m/cycle and that the final SIF ($K_{\text{max}}$) be smaller than the initial $K_{\text{max}}$ for the testing. Furthermore, to determine the load level during the different phases of pre-cycling, the Paris law was used. During the pre-cycling, the crack propagation was monitored periodically, and the load level was adjusted accordingly.

2.5.2. Static and fatigue tests

During the static tests, the beam was slowly loaded up to the maximum fatigue load ($F_{\text{max}}$) and then unloaded to $\sim 0.6$ kN. It was ensured that the beam complies with the conditions for linear elastic fracture mechanics (LEFM), which requires the length of the plastic zone ($r_p$) to be much smaller than the length of the crack ($r_p \ll a$). Static tests were performed before and immediately after strengthening, and after the fatigue test. For the fatigue test, the load ratio was set to $R = 0.1$. The loading frequency during the fatigue test was constant at 4.35 Hz, and the number of cycles was read directly from the digital counter on the testing machine.

3. Finite element modeling

Fig. 4 illustrates the finite element (FE) model of the strengthened beam. FE models were created using ABAQUS®/CEA 6.11 [25]. First, the un-strengthened beam with the notch and the crack at the mid-span was modeled. In the next step, the pre-stressed CFRP was...
added to the model. The assumptions and techniques used for the numerical model are presented below.

### 3.1. Material properties

The analysis was run in the linear elastic regime, so the material properties for the steel and CFRP were also defined as such. The steel was modeled as an isotropic linear-elastic material with Young’s modulus of 210 GPa and Poisson’s ratio of 0.3. The CFRP was modeled as an isotropic material. This is technically not correct as the composite nature of the laminate with the combination of the directionally arranged carbon fibers in the epoxy resin matrix results in orthotropic properties. However, because the CFRP in the model is only loaded in the fiber direction and is stress-free laterally, the assumption of isotropic material in the modeling, regardless of the orthotropic behavior of the CFRP, is assumed adequate. As previously studied at the Empa [26], the Young’s modulus is not perfectly linear for the CFRP material, and it increases slightly at higher stress levels. Therefore, different values were selected for the respective pre-stressing level.

<table>
<thead>
<tr>
<th>Specimens</th>
<th>Crack length [mm]</th>
<th>Pre-stressing level (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>B1</td>
<td>3</td>
<td>10</td>
</tr>
<tr>
<td>B2</td>
<td>20</td>
<td>20</td>
</tr>
<tr>
<td>B3</td>
<td>40</td>
<td>30</td>
</tr>
</tbody>
</table>

Fig. 3. Dimensions of the notch cut in the bottom flange and web of the steel beam.

Fig. 4. FE model of the CFRP-strengthened cracked steel beam.
3.2. Geometry, load, boundary conditions, and meshing

The specimen was modeled in accordance with the exact geometry of the test setup with solid general-purpose 20-node quadratic brick elements with reduced integration (C3D20R). Solid 3D elements were chosen over 2D shell elements as the calculation of SIFs via contour J-integrals is only possible for a 3D model in ABAQUS. To reduce the computational cost of the analysis, only half of the beam was modeled and a symmetry plane at the mid-span was added.

The support was modeled by tying the lower surface of the bottom flange in the region associated with a reference point using a rigid body constraint. The boundary conditions were then applied to the reference point to prevent any vertical and transversal displacement and rotations around the vertical and longitudinal axes. The displacement in the longitudinal direction of the support was not constrained, but the symmetry axis still prevented free body movements.

The load was applied as a pressure load only over the central 4.4 mm of the upper flange rather than the whole width of the flange to prevent longitudinal bending of the sides of the flange. For the strengthened beams, CFRP was added to the model. The anchorage clamps were modeled using a surface-to-surface tie constraint between the upper surface of the CFRP and the lower surface of the bottom flange. To prevent the CFRP from penetrating the bottom flange, a hard, frictionless surface-to-surface contact was created for the rest of the CFRP.

The specimen was meshed using a structured hex mesh with a global size of 20 mm for the steel beam and a global size of 15 mm for the CFRP. Owing to the stress concentration at the crack tip, the surrounding mesh over a rectangular area of 16 × 16 mm was refined to compute more accurate stresses and strains. The area was partitioned radially and circularly. The seeds were also assigned to each edge by defining either the element size or the number of elements (Fig. 4). The partitioned section was meshed using a radial sweep technique.

3.3. Modelling of the pre-stressed CFRP

The initial pre-stressing of the CFRP was modeled using a pre-defined stress field in the longitudinal direction. At the beginning of the analysis, the stress was homogenously distributed along the CFRP, and the load was not yet transferred onto the bottom flange of the beam through the tie constraint in the clamping region. During the first step of the analysis, no external load was added to allow for a redistribution of the stresses to reach a force equilibrium in the entire specimen. This results in reduction of the stress in the CFRP and an upward deflection of the whole beam. For the second load step, the external load was then defined.

3.4. Modelling of the crack

Compared with the recent advanced crack modeling techniques such as extended finite element method (XFEM) [27–29], a contour J-integral analysis was employed for the crack simulation. The J-integral method was deemed to be accurate enough for the present study. Initially, the crack front and the crack extension direction were defined. A value of 0.25 for the mid-side node was entered for the second-order mesh options and “Collapsed element side, single node” was chosen for the degenerate element control. This moves the mid-side nodes of the elements to the quarter-point and creates the singularity in the strain, which is needed to obtain accurate values for SIF during an elastic fracture mechanics analysis. A “History Output Request” was defined to write the results from the contour J-integral calculations into the output database. In all cases, at least 8 integrals were calculated for each time step and compared only once the results converged.

Fig. 5. Schematic view of the beam strengthened with CFRP system with a crack at the mid-span.
One problem that occurs in beams strengthened with pre-stressed CFRP is that the crack is initially under compression. This results in interpenetration of the crack faces, and ABAQUS gives negative values for the mode-I SIF, which is physically meaningless. To counteract this issue, the load application was split into two parts. During the first part, the load was applied until the crack started to open at the notch (F₀) with the symmetry plane defined for the entire cross-section. In the second step, the rest of the load was applied in addition to F₀, whereas the symmetry condition was changed to include only the un-cracked cross-section. While this is an adequate assumption in the context of this study, a better solution has already been developed that can model the crack surface contact correctly [30].

4. Analytical formulation

This section provides an analytical formulation for calculating the amount of pre-stressing needed to arrest the fatigue crack. The CFRP-strengthened beam system has 1 degree of indeterminacy and is statically indeterminate; thus, the internal force in the CFRP should be calculated using flexibility and energy methods. As can be observed in Fig. 5, the concentrated loads are applied at arbitrary and symmetric points (here C and C'), and the clamps are also located at arbitrary and symmetric points (here B and B'). The length of the pre-stressed CFRP after being clamped to the beam is determined as follows:

\[
\Delta L_P = \frac{PSL \cdot P_{CFRP}^* \cdot L}{100 E_{CFRP} A_{CFRP}^*} = \frac{PSL^* \sigma_{CFRP}^* \cdot L}{100 E_{CFRP}} \tag{1}
\]

where \( L \) is the length of the CFRP between the two clamps before applying any external force or pre-stressing (the length between B and B' in Fig. 5). \( L_i \) is the initial length of the CFRP after pre-stressing (i stands for the initial condition). \( A_{CFRP} \), \( E_{CFRP} \), \( \sigma_{CFRP} \), and \( P_{CFRP} \) are the cross-section, Young's modulus, ultimate tensile stress, and ultimate tensile force of the CFRP. \( PSL \) is the pre-stressing level in the CFRP. Using the geometry of the problem, the length after applying the external forces can be written as follows [26]:

\[
L_f = L + 2(c + h_0)\tan(\theta_0) - \int_0^L \frac{T}{E_0 A_{St}(X)} \, dX \tag{3}
\]

where \( L_f \) is the final length of the CFRP after applying external forces (f represents the final condition), \( c \) is the height of the clamps, and \( h_0 \) is half the height of the steel beam. \( E_0 \) and \( A_{St} \) are the Young's modulus and cross-sectional area of the steel beam, respectively. \( T \) is the force in the CFRP. The coordinate system is located at B for this part.

Using the compatibility, the following equation can be derived for the initial and final length of the CFRP:

\[
\Delta = \frac{TL}{E_{CFRP} A_{CFRP}^*} = L_f - L_i \tag{4}
\]

By using Eqs. (2) and (3) in Eq. (4) and after some simplifications, \( T \) can be derived as:

\[
T = \frac{F}{{E_{CFRP} A_{CFRP}^*}} \left[ \frac{c \tan(\theta_0)}{h_0} \left( \frac{2\pi}{\pi + \nu} \left( \frac{1}{l_{xw}} + \frac{h_1}{l_{xw}^2} \right) + \frac{1}{l_{xw}^2} \right) \right] + \Delta L_P \tag{5}
\]

Appendix A explains the detailed derivation of Eq. (5) and other assumptions made to derive the equation.

Ghafoori et al. [21] calculated the SIF for a cracked I beam strengthened with a pre-stressed CFRP and defined passive, semi-active, and active modes. Using their formulas and assuming the force to be derived by Eq. (5), the amount of pre-stressing needed to arrest a fatigue crack can be calculated as follow. The required pre-stressing was determined such that the mode-I SIF will be less than the threshold value of the utilized steel. Furthermore, in case that there was a mixed mode fatigue crack, i.e., \( K_{II} \neq 0 \), Hosseini et al. [31] presented a conservative design method to address the effects of mode II SIF on crack growth too. However, this will be out of the scope of the current study to analytically derive the formulation for the mix mode condition and further studies are required to numerically and analytically discuss the problem.

\[
PSL \geq \frac{100 E_{CFRP}}{\sigma_{CFRP}^* L} \left\{ \left( M_{max} \left( -\frac{1}{l_{xw}} + (\eta_1 + \eta_2) \right) \right) \frac{L}{E_{CFRP} A_{CFRP}^*} - \frac{2(e + h_0)\pi}{E_0 I_{St}(1 - \nu^2)} (\chi_1 + \langle h_0 \rangle) + \left( F^* \frac{c + h_0}{E_0 I_{St}} \left[ \frac{2\pi}{\pi + \nu} (l_1 + l_2) \chi_2 + \frac{l_1 (L - l_2)}{I_{xw} I_{xw}^2} \right] \right) \right\} \tag{6}
\]

To calculate the fatigue life of the structure, the modified Paris–Erdogan equation was used:
\[ N = \int dN = \int_{a}^{a_f} \frac{da}{C[\Delta K_{eff}]^m} \]  

(7)

where \( C \) and \( m \) are material constants (related to environmental conditions, stress ratio, and material properties), which can be found in different sources (e.g., British 7910 standard [32] and the Japanese society of steel structures handbook [33]). Because of the prestressing in the CFRP, the crack stays closed for most of the forces, so \( K_{op} \) will become zero and \( K_{max} \) will be equal to \( K_I \). Knowing this, \( \Delta K_{eff} \) can be expressed as follows:

\[ \Delta K_{eff} = K_{Icr}^{semi}(a,M,\rho_l) \]  

(8)

5. Results and discussions

5.1. Experimental results

5.1.1. Discussion on specimen B1

B1 is the specimen with a short crack of 3 mm from the tip of the notch and is strengthened with a CFRP pre-stressed to 10% of the ultimate tensile stress. A crack of 3 mm length was grown in a total of approximately 90,000 cycles. Initially, the lower load was set to 2 kN and the upper load to 4.5 kN. The crack was initiated after approximately 10,000 cycles and then grew 1.5 mm within 30,000 cycles. For the last 1.5 mm, the upper load was reduced to 3.7 kN to slow down the crack growth to \( 2 \times 10^{-5} \) mm/cycle. For the strengthening of B1, the CFRP was first pre-stressed up to 264.9 MPa (9.8% of the failure stress), and then the beam was placed on top of the pre-stressed CFRP and the clamps were tightened. After the actuator was unloaded, the stress in the CFRP was reduced by approximately 10% to 239.1 MPa. The result was an upward deflection of the beam, while the total stiffness of the system was more or less unchanged.

5.1.1.1. Fatigue tests. To determine the upper load level for the fatigue cycling, the beam was loaded statically after strengthening until the strain gauge next to the crack tip measured almost zero. Then, a linear regression line for the upper part of the curve was used to interpolate the curve and to find the point when the strain measurement turned from compression (negative values) to tension (positive values). This value corresponds to \( F_{ca} \) and is the point where the crack goes from semi-active to active (Fig. 6). Semi-active is a state of the crack when only part of the crack remains under compression. The crack becomes active when the crack tip is no longer under compression. Depending on the magnitude of the external load, pre-stressing force, and geometry, the stress distribution at the crack changes, which results in three crack modes: passive, semi-active, and active. More information on the states of the crack in the pre-stressed strengthened beam can be found in [20].

The upper fatigue load was then set to 6.0 kN, and the lower load was set to 0.6 kN, resulting in a load ratio of 0.1 for 1,000,000 fatigue cycles. The loading frequency was kept steady at 4.35 Hz and a periodic measurement was used to monitor the beam during cyclic loading. Fig. 7 shows the visual inspection of the crack initially and after 300,000, 700,000, and 1,000,000 cycles after which no crack propagation was observed. Additionally, no significant changes in the strain measurement next to the crack tip were observed, indicating that the crack remained dormant.

5.1.1.2. Fatigue loading with increased load. After 1,000,000 fatigue cycles were performed without any crack propagation, the load was increased just above the SIF threshold, \( \Delta K_{th} \), for 200,000 additional cycles. \( \Delta K_{th} \) was determined to be equal to 88.3 MPa√mm [34]. The increased load level was aimed at creating a crack propagation rate of \( 10^{-5} \) mm/cycle. There was a crack growth of 2 mm over

![Fig. 6. Strain measurement at the crack tip after strengthening of B1: calculation of \( F_{ca} \).](image-url)
the 200,000 cycles, which is very well in the visually observable range. Therefore, this corresponds to a $\Delta K_{\text{eff}}$ of 284.6 MPa$\sqrt{\text{mm}}$.

While the strain measurement next to the crack is able to identify the transition point where $K_I$ becomes zero, it cannot allocate certain strain values directly to the stress state at the crack tip. Therefore, the analytical model presented in Section 4 was used to back calculate the load level from the desired $K_I$ value. To use the model, the stress in the CFRP is required at the external load, where $K_I$ is calculated. Therefore, the strain measurement in the CFRP is used to determine the external load–CFRP stress relationship. For an external load of 6.1 kN and an axial stress in the CFRP of 266 MPa, the analytical model gives a $K_I$ value of $92.9$ MPa$\sqrt{\text{mm}}$, which is slightly lower than expected. As a result, the model is used to back calculate the external load corresponding to a value of 191.7 MPa$\sqrt{\text{mm}}$. This results in an upper load of 7.8 kN. The crack was again visually measured before and after cyclic loading. Fig. 8 shows the comparison of the initial crack before the fatigue cycling, after the 1,000,000 fatigue loads, and again after the 200,000 cycles with the increased load. The observed crack extension of approximately 1.7 mm corresponds to a propagation rate of $8.5 \times 10^{-6}$ mm/cycle.

5.1.2. Discussion on specimen B2

B2 was the specimen with a medium length crack of 20 mm from the tip of the notch and was strengthened with a CFRP pre-stressed to 20% of the ultimate tensile stress. The pre-crack for specimen B2 was grown over a total of approximately 275,000 cycles. The pre-cracking load and crack propagation for this test are summarized in Table 4. For the strengthening of B2, the CFRP was first pre-stressed up to 539.7 MPa (20% of the failure stress) before the beam was placed on top of the CFRP and the clamps were tightened. After release of the actuator, the stress in the CFRP was reduced to 463.7 MPa, equivalent to a reduction of 14%.

After pre-cracking up to 20 mm, a static test was performed. Owing to an operating error, the beam was loaded up to approximately 12 kN instead of 4 kN. This resulted in large plastic deformation and residual stresses around the crack tip. The estimation of the size of the plastic zone due to overloading was $r_p = 7.7$ mm. The residual strain measured by the strain gauge SG-3, which was positioned very close to the crack tip because the crack did not grow perfectly straight, was $424.8 \mu \text{m/m}$. Owing to the plastic deformations around the crack tip, the condition for LEFM is no longer valid and it has been observed that plastic overloading results in crack growth retardation effects [35]. Therefore, it was decided to grow the crack for an additional 8 mm up to a total of 28 mm from the notch in order to grow the crack tip out of the plastic zone. This removed the plasticity-induced effects from the crack tip and restored the LEFM conditions. Afterward, the strengthening and successive tests were continued as planned. The additional 8 mm crack length was grown.

![Fig. 7. Photographs of the crack after 0, 300,000, 700,000, and 1,000,000 cycles. A microscope with a build-in 8X magnifying glass was used to obtain the photographs.](image-url)
over 60,000 cycles with a load of 0.5–5 kN. For the strain measurement at the crack tip, an additional strain gauge was glued in the position next to the new crack tip location.

5.1.2.1. Fatigue tests. The load–strain measurement from the additionally placed strain gauge next to the crack tip is plotted in Fig. 9. The transition point when the crack tip of B2 goes from the semi-active to tension state and becomes active was observed at a load of 16.4 kN. Therefore, B2 was cyclically loaded between 1.7 and 16.3 kN for 1,000,000 fatigue cycles. While no crack propagation was observed visually, the strain measurement at the crack tip indicated some small changes during the fatigue loading, especially for the compressive strains that are largest at the unloaded part of the cyclic loading.

![Before fatigue test](image1)

![After fatigue test](image2)

![After fatigue test with increased load](image3)

**Fig. 8.** Size of the crack before and after fatigue loading. The photograph also shows the size of the crack after the load was increased (red arrow: reference point, yellow arrow: new crack tip). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

**Table 4**

Pre-cracking of specimen B2.

<table>
<thead>
<tr>
<th>Cycles [-]</th>
<th>Crack length [mm]</th>
<th>Lower load [kN]</th>
<th>Upper load [kN]</th>
<th>Crack propagating rate [mm/cycle]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0–25,000</td>
<td>0</td>
<td>0.6</td>
<td>5.0</td>
<td>–</td>
</tr>
<tr>
<td>25,000–160,000</td>
<td>0–5</td>
<td>0.6</td>
<td>5.0</td>
<td>$3.7 \times 10^{-5}$</td>
</tr>
<tr>
<td>160,000–230,000</td>
<td>5–18</td>
<td>0.6</td>
<td>6.0</td>
<td>$1.8 \times 10^{-4}$</td>
</tr>
<tr>
<td>230,000–275,000</td>
<td>18–20</td>
<td>0.6</td>
<td>4.0</td>
<td>$4.4 \times 10^{-5}$</td>
</tr>
</tbody>
</table>

![Graph](image4)

**Fig. 9.** Strain measurement at the crack tip after strengthening of B2: calculation of $F_{ca}$.

over 60,000 cycles with a load of 0.5–5 kN. For the strain measurement at the crack tip, an additional strain gauge was glued in the position next to the new crack tip location.

5.1.2.1. Fatigue tests. The load–strain measurement from the additionally placed strain gauge next to the crack tip is plotted in Fig. 9. The transition point when the crack tip of B2 goes from the semi-active to tension state and becomes active was observed at a load of 16.4 kN. Therefore, B2 was cyclically loaded between 1.7 and 16.3 kN for 1,000,000 fatigue cycles. While no crack propagation was observed visually, the strain measurement at the crack tip indicated some small changes during the fatigue loading, especially for the compressive strains that are largest at the unloaded part of the cyclic loading.
Afterwards, similar to the test of specimen B1, the upper load was raised to 17.7 kN, resulting in an increase of $K_I$ value in the analytical model to 285 MPa$\sqrt{\text{mm}}$, and the beam was loaded for 200,000 cycles, during which the crack grew by about 9 mm, as shown in Fig. 10.

5.1.3. Discussion on specimen B3

B3 was the specimen with a long crack of 40 mm from the tip of the notch and was strengthened with a CFRP pre-stressed to 30% of the ultimate tensile stress. The pre-crack for B3 was grown over a total of around 340,000 cycles, and the loads are summarized in Table 5. Compared to the first two beams with short and medium length cracks, the long crack substantially reduced the stiffness of the beam (Fig. 11).

For the strengthening of B3, the CFRP was first pre-stressed up to 813.5 MPa (30.1% of the ultimate tensile stress) before the beam was placed on top of the CFRP and the clamps were tightened. After releasing the actuator, the stress in the CFRP was reduced to 702.6 MPa.

5.1.3.1. Fatigue tests. The stress–strain measurement from the strain gauge next to the crack tip is plotted in Fig. 12. The transition point when the crack tip of B3 goes from the semi-active state to tension and becomes active was observed at a load of 21.5 kN. This resulted in a fatigue load of 2.1–21.5 kN for 1,000,000 cycles. No crack growth was observed, and the strain measurement at the crack tip did not indicate any propagation.

5.1.3.2. Fatigue loading with increased load. The increased load level for the 200,000 cycles was between 2.4 and 23.0 kN. Again, the upper load level was chosen to increase the $K_I$ value in the analytical model to 285 MPa$\sqrt{\text{mm}}$. During the 200,000 cycles, the crack grew by approximately 2 mm, and the strain measurement at the crack tip during the cyclic loading clearly indicated some changes, especially for the strain at the upper load level. A visual comparison of the crack before and after fatigue loading, and also after the increased load cycles, can be observed in Fig. 13.

5.2. Numerical and analytical outputs

Fig. 14 compares the experimentally measured mid-span deflections of specimen B1 at the two stages before and after strengthening with the results from the numerical analysis. The nonlinear effects for the un-strengthened beam are not present in the FE results, which show perfect linear-elastic behavior as expected. The deflection of the strengthened beams in the FE model is approximately 10% lower than the experimentally measured deflection, which is relatively good considering the assumptions made during the modeling regarding the boundary conditions and the strengthening. Similar to B1, a comparison of the FE model for B2 and B3 with the
experimental results was performed and showed a good agreement.

Fig. 15 compares the mode-I SIF calculated with the contour J-integral in ABAQUS and with the analytical model. They show good agreement with respect to the crack activation load as well as the slope afterwards. The plot also includes the crack activation load measured during the experimental test. The calculated mode-I SIF with the analytical model and the experimental crack activation load for B2 and B3 also agreed well.

Fig. 16 shows the SIF for a beam with a 74 mm crack length for various external loadings. By reducing the external loads, the stress in the CFRP becomes equal to the initial pre-stressing. Thus, both analytical methods, the conventional method suggested by Ghafoori et al. [20], which does not consider the effects of external loading on calculating pre-stressing and SIF and hereinafter is called the non-modified or conventional method, and the alternative method suggested in this paper, which is called the modified method, gives the same results. In addition, based on this figure, the crack activation moment for which the SIF becomes equal to zero increases and reaches 7.07 kN.m instead of its previous value, which was 6.2 kN.m. This shows that, as expected, external forces are acting in favor of

Table 5
Pre-cracking of specimen B3.

<table>
<thead>
<tr>
<th>Cycles [-]</th>
<th>Crack length [mm]</th>
<th>Lower load [kN]</th>
<th>Upper load [kN]</th>
<th>Crack propagating rate [mm/cycle]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0–40,000</td>
<td>0</td>
<td>1.0</td>
<td>5.0</td>
<td>–</td>
</tr>
<tr>
<td>40,000–65,000</td>
<td>0–1.5</td>
<td>1.0</td>
<td>5.0</td>
<td>$6.0 \times 10^{-5}$</td>
</tr>
<tr>
<td>65,000–245,000</td>
<td>1.5–37</td>
<td>1.0</td>
<td>6.0</td>
<td>$2.0 \times 10^{-4}$</td>
</tr>
<tr>
<td>245,000–340,000</td>
<td>37–40</td>
<td>0.7</td>
<td>4.2</td>
<td>$3.2 \times 10^{-5}$</td>
</tr>
</tbody>
</table>

Fig. 11. Load-deflection behavior of the three cracked beams.

Fig. 12. Strain measurement at the crack tip after strengthening of B3.
5.2.1. Calculation of the minimal pre-stressing

The FE analysis and analytical modeling could be useful for calculating the minimum pre-stressing needed to arrest the crack. The experimental plan could not so far determine the minimal pre-stressing level needed to arrest the crack at the same load level, which resulted in its propagation. Therefore, to address this need, a numerical model was used. For an upper load of 5 kN, the pre-stressing level needed for the crack tip to remain passive was computed. This load was selected as it results in a crack propagation rate of approximately $0.375 \times 10^{-8}$ mm/cycle for the un-strengthened beam depending on the crack length, as observed during the pre-cracking sections, and would result in failure relatively rapidly.

The analysis was run multiple times with increasing pre-stressing level (in increments of 5 MPa) in order to find the one that still left a very slight compression on the crack faces and no stress concentrations at the crack tip. With a pre-stressing stress of 195 MPa, the crack was fully open under a 5 kN external load, resulting in a stress concentration at the crack tip. When the pre-stressing was increased only slightly to 200 MPa, the crack tip remained passive for an external load of 5 kN. While the pre-stressing level could, in theory, be reduced slightly until $K_0$ is reached, this criterion was chosen instead for the comparison, as $K_0$ is dependent on multiple factors that are not taken into account in the model (e.g., load ratio, crack tip geometry, crack closure, and material properties). However, as long as the crack tip remains in the passive or semi-active state, the experimental results have shown that the crack will keep the crack closed and the analytical formula, which was suggested in this study, can predict this phenomenon correctly.
not propagate. Indeed, the defined passive and semi-active modes for a non-propagating crack conservatively guarantee the crack arrest, while eliminating the need for the identification of the material property. However, there exists cracked details where $K_{th}$ could not be ignored due to high achieved pre-stressing with respect to the conservative assumption. The results for all three beams are summarized in Table 6, which reveals that the minimal pre-stressing required is very similar for all three cases. It also indicates that a longer crack results in a slightly lower pre-stressing level.

The same analysis was also performed using the analytical model. The analytical model requires the actual stress in the CFRP with the external load applied as an input to precisely calculate the $K_I$ value. The experimental strain measurement on the CFRP for B1 and the numerical model with approximately 7% pre-stressing show that the stress lost due to the upward deflection is very similar to the stress regained when a 5 kN external load is applied. B1, for example, decreases from the initial value of 264.9 MPa to 239.1 MPa after the load is transferred onto the beam without external loading, and it goes back up to 260.1 MPa with a 5 kN external load. The change in the stress level depends on the stiffness of the specimen, but as the three beams have the same stiffness as long as the crack is under compression, the difference will be almost identical for all three cases. Therefore, it is reasonable to compare the loaded stress level used as the input in the analytical model with the initial pre-stressing level used as an input in the FE model. The results from the analytical model show very similar results to the numerical model, as can be observed in Fig. 17.

The analytical model was also verified with another series of experimental results to ensure that it can predict the prestressing for various systems. In a similar setup, a cracked I-beam pre-stressed with the PUR system was tested under fatigue loading. After pre-stressing the CFRP up to 30% of its ultimate strength, the loading range of 0.84 to 8.4 kN.m was applied to the specimen, with which the crack propagated from 12 to 74 mm. Fig. 18 shows the pre-stressing needed to arrest the crack under various loads for crack lengths of 12 and 74 mm. As observed from the figure, the difference between these two methods increases as the load increases, and this difference can be significant for real load conditions where the magnitude of the applied load is much larger than that in this experiment.
Using the Paris–Erdogan equation, the fatigue life of the beam with the existing crack was estimated using both the non-modified and modified analytical formulas. The crack length was assumed to increase from 12 to 74 mm, and the loading range was between 0.84 and 8.4 kN.m. Under this loading range, the crack goes under compression in each cycle, $K_{\text{min}}$ is assumed to be zero, and $ΔK_{\text{eff}}$ is equal to $K_{\text{max}}$ or the SIF calculated based on each method. Indeed, in practical cases, where relatively high pre-stressing is generated in the CFRP laminates, the stress ratio in the crack tip goes below zero, thus $K_{\text{I, min}}$ is assumed as zero. However, for conditions, where the pre-stressing is relatively low compared with the external loading (high loading ratio), the crack closure effects needs to be estimated to calculate the required pre-stressing [21]. Using these assumptions, the modified equation predicted 29,774 cycles in comparison with the non-modified method, which estimated 13,714 cycles. In the experiment, 37,050 cycles were necessary for the crack to propagate from 12 to 74 mm. It is obvious that the modified version can estimate the fatigue life of the structure much better because it considers the effects of the applied load on the prestressing, thus calculating a more precise SIF compared to that of the previous model. It is worth noting that the Paris–Erdogan equation can only be used for structures with pre-existing cracks and cannot be used to estimate the life of a structure before the initiation of the first crack.

6. Conclusions

Overall, this study concluded that the proposed strengthening system is an effective method to arrest a fatigue crack under pure bending as the stresses induced by the CFRP counteract the external bending moment and reduce the SIF of the crack tip. The following conclusions were drawn from the present study:

1. An experimental program for fatigue strengthening of a cracked I-beam with pre-stressed CFRP was designed. The tested specimens differed in crack length and applied pre-stressing level and allowed for an examination of this type of application within a useful range.
2. An analytical model was proposed, which was validated with experimental and numerical results showing good agreement. The results showed that the new analytical model can predict up to 2% lower prestressing compared to conventional models for small scale experimental beams. It is expected that this number increases significantly in real scale projects since the real scale external loads create more deflection in the beam, which then produces strain in the CFRP, resulting in increased prestressing in the CFRP composite.
3. It was shown that the crack did not propagate during fatigue cycles as long as the crack tip was under compression. It was also shown that the influence of the crack length on the required pre-stressing for crack arrest was negligible, and did not affect the crack activation load significantly. On the other hand, the pre-stressing level significantly influences the crack activation load.
4. As for the fatigue life prediction, it was shown that the current model was able to reduce the error in predicting the fatigue life by up to 40%, as a result of estimating a more precise value for the SIF in the CFRP member compared to the similar models.

<table>
<thead>
<tr>
<th>Crack length [mm]</th>
<th>Load [kN]</th>
<th>Pre-stressing level needed [%/MPa]</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>5</td>
<td>7.4/200</td>
</tr>
<tr>
<td>28</td>
<td>5</td>
<td>7.0/190</td>
</tr>
<tr>
<td>40</td>
<td>5</td>
<td>6.8/185</td>
</tr>
</tbody>
</table>

![Fig. 17. $K_I$ obtained from the analytical model at load of 5 kN for different pre-stressing levels.](image-url)
Appendix A

Assuming an oval crack in the web of the beam at the mid-span in Fig. 5, the integral in Eq. (3) can be simplified as follows (note that the coordinate system is moved to mid-span of the beam):

\[ x + \frac{L}{2} = X \Rightarrow dx = dX \tag{9} \]

\[ \int_0^L \frac{T}{E_s A_{st}(X)} dX = \frac{T}{E_{st}} \int_0^L \frac{dx}{A_{st}(x)} = \frac{T}{E_{st}} \phi \tag{10} \]

\[ \phi = \frac{\left(L - 2b\right)}{A_{st}} + 2 \int_0^L \frac{dx}{A_{1, st}(x)} + 2 \int_0^b \frac{dx}{A_{2, st}(x)} \tag{11} \]

where a and b are the depth and width of the crack, respectively and \( c \) is the distance between the centroid and the location where the crack intercepts the web. \( A_{st} \) is the un-cracked cross section of the steel beam, \( A_{1, st}(x) \) is the cracked section of the steel beam between 0 and \( c \), and \( A_{2, st}(x) \) is the cracked section of the steel beam between \( c \) and \( b \). Other parameters, such as \( t_f, b_f, t_w, \) and \( h \) are also shown in Fig. 5.

To find the rotation at B in Eq. (3), it was assumed that part BB’ acts like a simple supported beam. Based on the free body diagram.

---

**Declaration of Competing Interest**

None.
of part BB and assuming the effects of the removed parts, the rotation will be calculated as follows. The rotation at B can be written as:

$$\theta_b = \theta_b^\prime + \theta_b^*$$

(12)

where \(\theta_b^\prime\) is the rotation at B without the crack and \(\theta_b^*\) is the additional rotation in the beam due to cracking. This additional rotation shows that the cracked beam stores more energy compared to the un-cracked beam. This extra potential energy is called \(U_T\) and is stored near the crack itself. Based on Castigliano’s second theorem, this extra rotation can be calculated by taking the derivative with respect to the moment at the end of the beam [36]. Because of symmetry, the equation can be expressed as:

$$\theta_b^\prime = \frac{1}{2} \frac{\partial U_T}{\partial M_b}$$

(13)

Knowing the Paris law, \(U_T\) can be expressed as:

$$U_T = \int_{A_c} J_i(a) da$$

(14)

where \(A_c\) is the crack surface and \(J_i(a)\) is the potential energy release rate, which for the problem of this study can be derived using the following equation:

$$J_i(a) = \frac{1}{E'} \left[ \left( \sum_{i=1}^{6} K_i \right)^2 + \left( \sum_{i=1}^{6} K_{II} \right)^2 + m \left( \sum_{i=1}^{6} K_{III} \right)^2 \right]$$

(15)

where \(E = E_s\) when assuming the plane stress condition applied and \(E' = E_s/(1 - \nu^2)\) in the case of plane strain condition. \(\nu\) is Poisson’s ratio of steel. For this problem, a plane stress condition was applied. \(m\) is a factor that is equal to 1 + \(\nu\) for the plane stress condition and 1/(1 - \(\nu^2\)) for the plane strain condition. \(K_i, K_{II},\) and \(K_{III}\) are SIFs for mode I, II, and III fracture, respectively, which must be calculated for different loading cases, i, separately. Owing to the loading and geometry of the problem at hand, \(K_i = K_{II} = 0\) for all \(i\) values. \(K_i\) is calculated using the G* integral and energy release rate. Since the notch was created at the mid-span length of the beam tested under four-point bending, the crack grew perpendicular to the cross-sectional area vector and in a vertical and straight direction. At the mid-span, material experiences pure bending with no shear; thus, the highest tensile stress that causes the failure exists in the aforementioned plane. This can be seen by investigating the Mohr-Coulomb diagram of the points under pure bending. Experimental results also showed an almost straight crack path. Therefore, it can be said that the proposed model properly follows a real crack path of the case that was investigated.

In the case of three-point bending and if the crack is located anywhere except the mid-span or if the material has some defects at some points that direct the failure to that direction, the crack will no longer be straight and will follow a curvilinear path. In this case, the crack will be under both tension and shear; thus, \(K_i\) cannot be assumed to be equal to zero anymore. In this case, the critical direction can be found by drawing the Mohr-Coulomb diagram for each point according to the state of stresses at that point. This study only investigated the crack growth under the given circumstances and the derived equations need to be modified for curvilinear cracks according to the location of the cracks.

The SIF of cracked steel I-beams calculated by Ghafoori and Motavalli [20] was used to determine the SIF for mode I.

$$K_i = 2 \left( - \frac{T^2}{A_s} - \frac{(M_k - Y_s T)^2}{I_{Z,St}} + T^2 (\lambda_1 + \lambda_2) + M_k^2 \eta_1 + \eta_2 \right) \frac{\pi}{8} \left( 1 - \nu^2 \right)^{\frac{1}{2}}$$

(16)

where \(T\) is the axial force, \(M_k\) is the moment at the cracked section, \(A_s\) and \(I_{Z,St}\) are the area and second moment of inertia of the steel I beam’s un-cracked section, respectively, and \(Y_s, \lambda_1, \lambda_2, \eta_1,\) and \(\eta_2\) are parameters related to the geometry of the beam and are discussed in detail in reference [21]. Using Eq. (16), the SIF due to the flexural moment and axial force were calculated and added to find \(J_i(a)\). Using the chain rule and knowing that \(\partial M_k/\partial M_b = 1\), Eq. (13) (or \(\theta_b^\prime\)) is solved as:

$$\theta_b^\prime = \frac{1}{2} \int_{A_c} \frac{\partial J_i(a)}{\partial M_k} \frac{\partial M_k}{\partial M_b} da = \frac{\pi}{8} \left( 1 - \nu^2 \right)^{\frac{1}{2}} \left[ - T(\chi_1 + (c + h_b) \chi_2 + \chi_3) + F(l_1 + l_2) \chi_2 \right]$$

(17)

$$M_c = M_s + F(l_2) - T(Y_c)$$

(18)

$$\chi_1 = \int_{A_c} \left( - \frac{1}{I_{Z,St}} + (\eta_1 + \eta_2) \right) \left( - \frac{1}{A_s} + (\lambda_1 + \lambda_2) + Y_s^2 (\eta_1 + \eta_2) \right)^{\frac{1}{2}} da$$

(19)

$$\chi_2 = \int_{A_c} \left( - \frac{1}{I_{Z,St}} + (\eta_1 + \eta_2) \right) \left( - \frac{1}{A_s} + (\lambda_1 + \lambda_2) + Y_s^2 (\eta_1 + \eta_2) \right)^{\frac{1}{2}} da$$

(20)

$$\chi_3 = \int_{A_c} Y_c \left( - \frac{1}{I_{Z,St}} + (\eta_1 + \eta_2) \right) \left( - \frac{1}{A_s} + (\lambda_1 + \lambda_2) + Y_s^2 (\eta_1 + \eta_2) \right)^{\frac{1}{2}} da$$

(21)

where \(l_1\) and \(l_2\) are the lengths of different segments of the beam and are defined in Fig. 5. The rotation for the simply supported un-cracked beam can also be derived using basic structure knowledge. Finally, by using the results obtained from Eqs. (17) and (10) in Eq.
the force in the CFRP can be calculated according to Eq. (5).

References