Fatigue damage measures with a statistical model for the Wöhler field

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ABSTRACT: This paper deals with the problem of accumulating damage due to any fatigue load history. First, some desirable properties for a damage model are discussed and different possible alternatives fulfilling these properties are analyzed. Generally, current damage measures, such as the Palmgren-Miner's rule, do not satisfy these properties. Next, a statistical fatigue regression model, able to predict the Wöhler field for any combination of \( \sigma_{\text{min}} \), \( \sigma_{\text{max}} \) or \( R = \sigma_{\text{min}}/\sigma_{\text{max}} \) is presented. This model is based on physical, statistical and compatibility conditions rather than on arbitrary functions. The probability of failure is assumed to be the most suitable option to assess damage measure. According to this, the procedure for obtaining the basic information for constant load, i.e. the probabilistic Wöhler field, is discussed, and formulas for calculating the associated damage are given. Finally, the proposed damage accumulation approach is applied to failure prediction of pieces under a Gaussian and FALSTAFF load spectra.

1 INTRODUCTION AND MOTIVATION

The evaluation of fatigue damage is basic and very important in design of structures, because fatigue is becoming determinant and the cause of a high number of failures of mechanical components and structural elements in real practice.

Laboratory tests are usually carried out under constant stress ranges and levels (S-N curves) or under a particular accelerated load history (ASTM E606), but real structures are subjected to fatigue load histories involving much more complex varying stress ranges and stress levels.

There is a long list of models dealing with the parametric definition of the S-N curves for a constant reference stress level, however, for these models to be applicable to real situations, they must be complemented with a damage accumulation model to allow engineers the evaluation of fatigue damage. The complex load history furthermore needs a counting method (ASTM E1049) and other transformations of the real load history to get an equivalent simplified load spectrum. The mean stress must be considered as a secondary parameter besides the stress range, as the main parameter. Models based on empirical assumptions have been already proposed (Collins, 1993).

An important question was risen in Castillo et al. (Castillo et al., 2007c): what is damage? Or, equivalently how is damage related to failure? A physical concept of failure, such as crack size, seems to be adequate to define a service limit state, but fatigue failure, as an ultimate limit state, requires a probabilistic framework allowing relating damage levels to probabilities of occurrence. The solution to these question is one of the principal objectives of this paper. Furthermore, several methods to measure damage will be analyzed and discuss to identify which is the best one and which of them are inadequate for measuring damage.

2 THE GUMBEL-WÖHLER FIELD

Consider a fatigue test with an alternating constant load, from minimum, \( \sigma_m \), to maximum, \( \sigma_M \), stresses, respectively. Castillo et al. (Castillo et al., 2007a) determine the cumulative distribution function of the random lifetime \( N \) (number of cycles to failure) associated with the test, a Gumbel family of models able to reproduce not only the whole Wöhler field for \( \sigma_M \) constant and varying \( \sigma_m \), but the whole Wöhler field for \( \sigma_m \) constant and varying \( \sigma_M \).
The proposed Gumbel model is:

\[
p = 1 - \exp \left\{ - \exp \left[ \frac{(\log N^* - B)(\log \Delta \sigma^* - C) - E}{D} \right] \right\}
\]  

(1)

where \(0 \leq p \leq 1\) refers to the percentile, \(\Delta \sigma^* = \sigma_M^* - \sigma_{M0}^*, \sigma_M^* = \sigma_M/\sigma_0, N^* = N/N_0\) and \(\sigma_0\) and \(N_0\) are some reference stress and number of cycles, respectively, to make the formulas dimensionless.

The physical meanings of the parameters are: \(A\) is the Weibull shape parameter of the cumulative distribution function (cdf) in the S-N field, \(B\) is the threshold value of log-lifetime, \(C\) is the endurance limit, \(E\) is the parameter defining the position of the corresponding zero-percentile hyperbola and \(D\) is the Weibull scale factor.

Since the model must be valid for any fixed values of \(C\) and \(\Delta \sigma\), and because for different constant values of \(\Delta \sigma\) one must have different models of the form (1), the parameters \(A, B, C, D\) and \(E\) must be functions of \(\sigma_m\). Similarly, if the constant load fatigue tests are run for constant values of \(C\), one has another family of models, where now the parameters \(A, B, C, D\) and \(E\) are functions of \(\sigma_m\).

In order all these functions to be compatible, the final model must be of the form (Castillo et al., 2006), (Castillo et al., 2008), (Castillo et al., 2007b), (Koller et al., 2007b) and (Koller et al., 2007a):

\[
F(N) = 1 - \exp \left\{ - \exp \left[ \left( C_0 + C_1 \sigma_m^* + C_2 \sigma_M^* + C_3 \sigma_m^* \sigma_M^* + (C_4 + C_5 \sigma_m^* + C_6 \sigma_M^* + C_7 \sigma_m^* \sigma_M^*) \log N^* \right) \right] \right\}
\]  

(2)

where \(F\) is the CDF function of \(N^*\), subject to:

\[
C_4 \geq 0
\]  

(3)

\[
C_3 - C_4 \leq 0
\]  

(4)

\[ -C_3 - C_4 \leq 0 \]  

(5)

\[-C_4 + C_3 - C_6 + C_7 \leq 0 \]  

(6)

\[-C_4 - C_3 - C_6 - C_7 \leq 0 \]  

(7)

\[ C_1 C_6 - C_2 C_7 \geq 0 \]  

(8)

\[ C_3 C_5 - C_1 C_7 \geq 0 \]  

(9)

\[ C_6 (C_6 + C_7) - C_7 C_8 \leq 0 \]  

(10)

\[ C_7 (C_7 + C_8) - C_7 C_8 \leq 0 \]  

(11)

where \(\gamma = 0.57772\) is the Euler-Mascheroni number. Constraints (3)-(11) come from the following physical conditions:

- The asymptotic value \(\Delta \sigma_m\) or \(\Delta \sigma_M\) must be non-negative, and, due to physical reasons, must be non-increasing in \(\sigma_m\) or \(\sigma_M\) respectively.
- The cdf in (2) must be non-decreasing in \(\log N\), non-increasing in \(\sigma_m\) and non-decreasing in \(\sigma_M\).
- The curvature of the zero-percentile of \((\log N, \sigma)\) for constant \(\sigma_m\) must be nonnegative, and, in the case of constant \(\sigma_M\) must be non-negative.

We remind the reader that the cumulative distribution function of the two parameter Gumbel family is given by:

\[
F(x; \lambda, \delta) = 1 - \exp \left\{ - \exp \left[ \frac{x - \lambda}{\delta} \right] \right\}; \quad x \geq \lambda
\]  

(12)

Where \(\lambda\) and \(\delta\) are the location and the scale parameters, respectively. When \(X\) follows a Gumbel distribution \(G(x; \lambda, \delta)\), we write \(X \sim G(x; \lambda, \delta)\), its mean \(\mu = \lambda = 0.577728\) and variance \(\sigma^2 = \pi^2\delta^2/6\).

One important property of the Gumbel family is that it is stable with respect to location and scale transformations, and also with respect to minimum operations (Castillo, 1988) and (Castillo et al., 2004). More precisely:

\[
X \sim G(\lambda, \delta) \iff \frac{X - a}{b} \sim G \left( \frac{\lambda - a}{b}, \frac{\delta}{b} \right)
\]  

(13)
and
\[ X_i \sim G(\lambda, \delta) \Leftrightarrow \text{Min}(X_1, X_2, \ldots, X_n) \sim G(\lambda - \delta n, \delta) \] (14)

2.1 Parameter estimation

The parameter estimation of the model can be done by several methods. The most well known method for estimating the parameters of a statistical model is the maximum likelihood method, which shows good statistical properties (Castillo et al., 2008).

Other estimation methods, can be seen in (Castillo & Hadi, 1995) and (Castillo et al., 1999).

2.2 Normalization of the Gumbel model

Normalization will be applied with the aim of establishing a relation among the fatigue data pertaining to different load levels, thus enabling us not only a pooled parameter estimation, but the statistical interpretation of the damage measure to be done.

According to Eq. (13), it is generally accepted in the case of fatigue lives, \( U = (x - \mu_x)/\sigma_x \) (corresponding normalized value) also follows a Gumbel distribution with parameters \( \lambda^* \) and \( \delta^* \), given by:

\[
\lambda^* = \frac{\lambda - \mu}{\sigma} = \frac{0.57722}{\pi}; \quad \delta^* = \frac{\delta}{\sigma} = \frac{\sqrt{6}}{\pi} \quad (15)
\]

Thus, it follows, that all distributions sharing common parameters, and \( \pm \) transform, after normalization, to the same distribution.

An alternative normalization consists of using the random variable

\[
Z = \frac{(\log N^* - B)(\log \Delta \sigma^* - C) - E}{D} \quad (16)
\]

3 DAMAGE MEASURES

The problem of defining and selecting damage measures must be done with care if one desires they to be useful in practice. This problem is discussed in this section, with the main ideas taken from Castillo et al. (Castillo et al., 2007c).

3.1 Some requirements for damage measures

Some properties to get a valid damage measure are:

- **Property 1.** Increasing with damage: The larger the damage, the larger must be the value of the damage measure.
- **Property 2.** Interpretability: The damage measures must provide a clear information of the associated damage level.
- **Property 3.** Dimensionless measures: The use of dimensions causes problems in the estimation of lifetime and requires indication of the measure units used in the analysis. So, dimensionless measures are much more convenient.

- **Property 4.** Known and fixed range: The range of variation of the damage measure must be fixed and known, independently of the type of load and, if possible, of the material.
- **Property 5.** Of known distribution: To know the probability of failure of a piece chosen at random, its damage must have a known distribution.

3.2 Some damage measures

Measures based on the number of cycles: As fatigue damage increases with the number of cycles \( N \), the number of cycles to failure or any increasing function of it are possible candidates for damage measures.

1. **The number of cycles:** The damage measure is the number of cycles \( N \). This measure does not satisfy Property 3 above (is not dimensionless), because if we use thousands or dozens of cycles, instead of cycles, we have \( N/1000 \) or \( N/12 \), respectively.

2. **The logarithm of the number of cycles:** Other alternative is the logarithm of the number of cycles \( N \). Unfortunately, this index satisfies neither Property 3 nor Property 4.

3. **The normalized logarithm of the number of cycles:** A third alternative is the normalized logarithm of the number of cycles to failure.

\[
\mu_i \quad (17)
\]

where \( \mu_i \) is the mean value of the number of cycles to failure \( N_i \) for a stress level \( \Delta \sigma^* \).

4. **The standardized logarithm of the number of cycles:** The last option analyzed, is the standardized logarithm of the number of cycles:

\[
N^* = \frac{\log N_i - \mu_i}{\sigma_i} \quad (18)
\]

This alternative does not depend on the stress range and material, so we can conclude that \( N^*_i \) does not satisfy all the properties.

**Based on the Palmgren-Miner number:**

5. **The Palmgren-Miner number:**

\[
M_i = N_i/\mu_i \quad (19)
\]

where \( \mu_i \) is the mean value of the number of cycles to failure \( N_i \) for a stress level \( \Delta \sigma^* \).
The Palmgren-Miner number is dimensionless (because the random variable is divided by its mean) so it satisfies Property 3. Taking logarithms one gets:

$$\log M_i = \log N_i - \log \mu_i'$$  \hspace{1cm} (20)

Again, the range of $M_i$ depend of the material, so it does not satisfy Property 4.

6. The logarithm the Palmgren-Miner number: Similarly to the previous measure, the range does not satisfy Property 4.

Based on the Gumbel variable: In Section 2.2 (eq. (16)), a very convenient candidate for a damage measure, the normalized variable $Z$ was presented. It satisfies all the Properties: increasing with damage, interpretable, non-dimensional, with fixed range, and $Z \sim G(0, 1)$. Thus, we propose $Z$ as a convenient measure for the cumulative fatigue damage associated with a given stress level or load history.

Based on the failure probability: Above the zero-percentile curve, we can define our damage measure as the failure probability (Castillo et al., 2007c):

$$P_F = F(U(B, -\lambda_i, \delta)) = F_D(D)$$  \hspace{1cm} (21)

where $P_F$ is non-dimensional and has a uniform distribution ($P_F \sim U(0, 1)$). This measure satisfies all desired Properties 1-5.

To define a damage measure below the zero percentile, in the non-failure zone, a proportionality criterion is used. This criteria was defined in (Castillo et al., 2007c), damage in this zone is proportional to the number of cycles if the stress level is held constant.

Critical comparison: A comparison between all the cases studied is made. Table 1 summarizes the main characteristics and properties of several proposals of damage measures.

The main conclusion derived from this table is only measures $Z$ and $P_F$ satisfy all the properties, so these measures are the best, followed by $N^*$.

3.3 Damage accumulation

Model (2) provides probabilistic bases for calculating the damage accumulation for any type of loading being considered. In fact, due to the possible identification of the probability of failure $P_F$ represented by the percentile curves in the Wöhler field for any $\sigma_{\text{min}}$ and $\sigma_{\text{max}}$ with the damage state (Castillo et al., 2007c), the model can be used in cumulative damage calculations for fatigue life prediction of components subject to complex loading histories. To evaluate the accumulated damage, we proceed as shown in (Koller et al., 2007b).

4 EXAMPLE OF APPLICATION

As indicated, this model provides probabilistic bases for calculating the damage accumulation for any type of loading being considered. In fact, due to the possible identification of the probability of failure $p$, represented by the percentile curves in the Wöhler field, with any damage state (Castillo et al., 2007c), the model can be used in cumulative damage calculations for fatigue life prediction of components subject to complex loading histories.

4.1 Parameter estimation

To characterize the material, different load fatigue tests were conducted with steel 42CrMo4. The static characteristics of the material were $R_m = 1067$ MPa and $R_{\text{op}} = 967.3$ MPa.

<table>
<thead>
<tr>
<th>Damage measure</th>
<th>Increasing</th>
<th>Interpretable</th>
<th>Dimensionless</th>
<th>Range</th>
<th>Distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_i$</td>
<td>Yes</td>
<td>**</td>
<td>No</td>
<td>$(e^\lambda, \infty)$</td>
<td>$\log G(B - \lambda_i, \delta_i)$</td>
</tr>
<tr>
<td>log $N_i$</td>
<td>Yes</td>
<td>*</td>
<td>No</td>
<td>$(\lambda_i, \infty)$</td>
<td>$G(B - \lambda_i, \delta_i)$</td>
</tr>
<tr>
<td>$D_i = \log N_i/\mu_i'$</td>
<td>Yes</td>
<td>*</td>
<td>Yes</td>
<td>$(\lambda_i/\mu_i, \infty)$</td>
<td>$G(\lambda_i/\mu_i, \delta_i/\mu_i)$</td>
</tr>
<tr>
<td>$N^* = (\log N_i^* - \mu_i)/\eta_i$</td>
<td>Yes</td>
<td>***</td>
<td>Yes</td>
<td>$(\lambda_i', \infty)$</td>
<td>$G(\lambda_i', \delta_i')$</td>
</tr>
<tr>
<td>$M_i = N_i/\mu_i$</td>
<td>Yes</td>
<td>**</td>
<td>Yes</td>
<td>$(e^{\lambda_i'/\mu_i}, \infty)$</td>
<td>$\log G(\lambda_i - \log \mu_i', \delta_i)$</td>
</tr>
<tr>
<td>log $M_i = \log N_i - \log \mu_i'$</td>
<td>Yes</td>
<td>*</td>
<td>Yes</td>
<td>$(\lambda_i - \log \mu_i', \infty)$</td>
<td>$G(\lambda_i - \log \mu_i', \delta_i)$</td>
</tr>
<tr>
<td>$Z = (\log N^* - B - \lambda_i)/\delta_i$</td>
<td>Yes</td>
<td>***</td>
<td>Yes</td>
<td>$(-\infty, \infty)$</td>
<td>$G(0, 1)$</td>
</tr>
<tr>
<td>$P = F(\log N_i)$</td>
<td>Yes</td>
<td>****</td>
<td>Yes</td>
<td>(0, 1)</td>
<td>$U(0, 1)$</td>
</tr>
</tbody>
</table>

$\lambda^* = 0.57772\sqrt{6}/\pi; \delta^* = \sqrt{6}/\pi$
The test specimens (Figure 2) were cylindrical of 8 mm diameter. The total and free lengths were 130 mm and 30 mm, respectively, and the transition radius to the test section of the specimen was 55 mm.

All the tests were conducted as shown in (Koller et al., 2007a) and (Koller et al., 2007b). The test data are shown in these references.

With all these data, the model parameters were estimated using the maximum likelihood method. The resulting model is:

\[
F(N) = 1 - \exp \left( - \exp \left[ -53.21712 - 33.572Ia: + 47.3862atJ + 24.9346a:2 + (0.5102(1 - \sigma_m^*) + 0.04390-tJ + 0.023Ia-:,a-tJ) \log N^* \right] \right)
\]

(22)

The damage probabilities calculated from (2) were obtained, using these parameters. To test the goodness of the model, the Kolmogorov-Smirnov and chi-square uniformity tests were applied (Koller et al., 2007a).

4.2 Damage evaluation

To check the validity of the proposed model for the calculation of the accumulated damage for failure prediction, different load histories have been used:

- Constant load history: Described by $\Delta \sigma^*_1 = 1130$ MPa.
- Linear variable load history: Two different cases are studied, $\Delta \sigma^*_1 = 300 - 5 \cdot 10^{-3} N \text{ MPa}$ (dashed line) and (c) linearly decreasing: $\Delta \sigma^*_1 = 1000 + 10^{-3} N \text{ MPa}$ (dotted line).

Figure 3 shows the resulting accumulated values of damage obtained from the model with the parameters for the three loading types cited before. It can be appreciated how the model is capable of evaluating the cumulative damage of different types of loads. This Figure illustrates how the damage $P(N)$ increases with the number of cycles, and with this model it would be easy to fix a probability of failure, say $10^{-4}$, and then calculate the associated design number of cycles.

Figure 4 shows the $P_F$ damage curves resulting from the model for both spectra. In this Figure, it can be observed how from this model the damage or failure probability of the sample is null or very low during the first cycles. Later, the damage increases until it approaches one. One of the main advantages of this model is that it permits to know the probability range in which a sample will break, so that an engineer could know with precision the number of cycles below which the failure probability of the sample is sufficiently low.
5 CONCLUSIONS

The main conclusions from this paper are the following:

1. The normalized random variable in Eq. (16) described in Section 2 appears to be a useful tool to facilitate the comparison of the cumulative fatigue damage produced by different load histories involving constant or changing stress levels.

2. A wide range of possible alternatives for selecting damage indices, including the logarithm of the number of cycles, the number of cycles to failure, the Palmgren- Miner number, its logarithm, its normalized or standardized form, the reference Weibull variable and the failure probability, are possible, and have been described in this paper. However, some of them are more convenient than others in the sense that they satisfy some desirable properties, such as adimensionality, fixed range, interpretability, known distribution, invariance with respect to load histories, etc.

3. The probability of failure is a very reasonable criterion for defining cumulative damage associated with different load histories. In fact, the Wöhler percentile curves allow us an easy interpretation of damage.

4. The new model (2) is a good method for evaluating the damage in all ranges and load histories.

5. Finally, it can be concluded that the model can be used for any type of load, constant, Gaussian or other. In our study we could observe that the model is very sensitive to the parameters used so, the data used for estimating these parameters have to be carefully chosen. Anyway, more research is needed about this subject.

REFERENCES


