Localized quasiholes and the Majorana fermion in fractional quantum Hall state at $\nu = \frac{5}{2}$ via direct diagonalization

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Using exact diagonalization in the spherical geometry, we investigate systems of localized quasiholes at $\nu = 5/2$ for interactions interpolating between the pure Coulomb and the three-body interaction for which the Moore-Read state is the exact ground state. We show that the charge $e/4$ quasihole can be easily localized by means of a δ -function pinning potential. Using a tuned smooth pinning potential, the quasihole radius can be limited to approximately three magnetic length units. For systems of two quasiholes, adiabatic continuity between the Moore-Read and the Coulomb limit holds for the ground state, while for four quasiholes, the lowest two energy states exhibit adiabatic continuity. This implies the existence of a Majorana fermion for pure Coulomb interaction. We also present preliminary results in the Coulomb limit for braiding in systems containing four quasiholes, with up to 14 electrons, diagonalizing in the full spin-polarized sector of the second Landau-level Hilbert space.

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I. INTRODUCTION

The possibility of realizing non-Abelian braiding statistics¹ in a condensed-matter system has generated a great deal of interest both for theoretical and experimental studies. One such system that may be closest to experimental realization is the fractional quantum Hall (FQH) state at filling fraction $\nu = 5/2$, first observed by Willett *et al.*² Whether the experimentally observed $5/2$ state indeed has charged excitations with non-Abelian braiding statistics is still unknown, although the discovery of a neutral current in experiment may make us hopeful.³ For a review, we refer the reader to Nayak *et al.*⁴ and to Stern.⁵

In an earlier work⁶ we presented numerical evidence that the FQH ground state (GS) at $\nu = 5/2$ for Coulomb interaction, when spin polarized, is in the same universality class as the non-Abelian Moore-Read (MR) Pfaffian state.⁷ Calculating the energy spectrum of the few lowest-energy states by exact diagonalization for electron interactions interpolating between the Coulomb interaction V_C and the three-body interaction V_{3b} , for which the MR state is the unique GS, we found for all examined system sizes adiabatic continuity (AC) of the GS and no sign of a decrease of the gap between the two limits V_C and V_{3b} . We concluded that AC can be expected even in the thermodynamic limit. In addition, we drew a phase diagram in the two-body interaction space (in the vicinity of the Coulomb interaction), showing that the FQH gapped phase coincides with the MR phase.

Previous theoretical work^{8–10} gave indications for AC for the spin-polarized $\nu = 5/2$ GS between the MR and the Coulomb limit. In the disk geometry Xin Wan *et al.*⁹ were also able to localize a single quasihole (QH) at the center of the disk showing that, in a certain range of QH pinning and system confinement potential strength, the lowest-lying state belongs to the total angular momentum value that corresponds to a MR $e/4$ charged QH, for electron interactions interpolating between V_{3b} and V_C .

In spite of the AC of the GS, the question whether the elementary charged excitations preserve their non-Abelian

properties in going from the MR to the Coulomb limit still needs to be examined: This is a very important issue, because braiding of non-Abelian quasiparticles has been proposed for topological quantum computation.^{4,11} At $\nu = 5/2$, one expects that QHs can have either charge $e/4$ with the possibility of having non-Abelian braiding statistics or $e/2$ with Abelian fractional statistics. Indeed, the results by Tóke *et al.*¹² have cast doubt on the existence of localized QHs with charge $e/4$.

Here, we study fully spin-polarized systems with localized QHs in the spherical geometry. This polarization choice is motivated by theoretical investigations concerning this issue: One of us showed that the GS of the disorder-free FQH state at $\nu = 5/2$ is spin polarized even for vanishing Zeeman energy;¹³ recent theoretical work^{14,15} confirmed this result. However, from the experimental point of view, the situation is less clear. Transport experiments with variable electron density^{16,17} or in a tilted magnetic field at high density¹⁸ have been interpreted in controversial ways: Varying the electron density by more than a factor 2,¹⁶ the excitation gap varies smoothly with no break in slope or discontinuity, indicating that neither GS nor excitations change their character while the Zeeman energy varies by an amount that exceeds the measured energy gap by a factor of 5–10. That an unpolarized state would survive under this condition appears unlikely. Also, in high-density samples, the disappearance of the gapped phase in a tilted magnetic field cannot be explained by a Zeeman energy term with a GaAs bulk g factor.¹⁸ Nevertheless, arguments have been presented that these experimental results might be consistent with a spin-unpolarized $5/2$ state.¹⁹ Also first results from direct spin polarization measurements, using optical methods,^{20,21} appear to be consistent with a spin-unpolarized state. As the experimental samples are likely to be inhomogeneous and consist of compressible regions separated by percolating incompressible filaments at $\nu = 5/2$,²² it is not clear that the optical data relate solely to the filaments or to the compressible regions.²¹ The separate determination of the polarization state of the incompressible parts has not yet been possible. In addition, charged impurities may also induce skyrmions in the incompressible domains,^{21,23} which

will decrease the total spin polarization. We also note that, once thermally assisted tunneling is taken into account,²² the gap obtained from analyzing the dissipative conductance can be as consistent with expectations for a spin-polarized system at $\nu = 5/2$ (Refs. 17,24 and 25) as for the hierarchy state at $\nu = 4/9$.

In our investigations at first we show that charge $e/4$ QHs can be localized by a δ -function potential and, varying the form of the localization potential away from the simple δ function, we also show that there is an optimal localizing potential that minimizes the density oscillations around the QHs. We also find that in systems with two QHs, in the Coulomb case, it is sufficient to have a single δ function: This localizes a single QH in the GS, while the other is automatically at the antipodal point on the sphere, because of the Coulomb repulsion.

We then study how states containing two and four QHs at $5/2$ evolve as the interaction is varied between V_{3b} and V_C , and show that there is AC between the two interaction limits, as in the previous work.⁶

For two QHs, as the interaction is varied from V_{3b} to V_C , we find that the lowest-energy state does not mix with the higher lying ones. For systems containing four QHs, the two lowest-lying states forming a degenerate doublet in the V_{3b} limit^{26,27} remain the two lowest-energy states even in the limit of pure Coulomb interaction V_C . Thus, the Majorana fermion associated with the doublet²⁸ appears to survive in the Coulomb limit.

To better study the MR limit we then repeat the investigations for four QHs using a different pinning potential: Also with this approach we obtain AC for the lowest-lying doublet between the two interaction limits, confirming the above results. Finally we present some preliminary results of braiding with QHs, showing that exchanging the position of two QHs, in the presence of two others fixed QHs, the system goes from one of the MR doublet states to the other, showing non-Abelian braiding statistics. Furthermore, we investigate the fusion of two QHs, getting an estimate for the MR doublet splitting.²⁹

II. THE SYSTEM

In the following we consider fully spin-polarized electron systems on the surface of a sphere:³⁰ By exact diagonalization we obtain their low-lying states and energy spectra. In the spherical geometry for a half-filled Landau level (LL) the number of electrons N and the number of flux quanta N_ϕ are related by $N_\phi = 2N - S$, where the shift S (Ref. 31) is a topological quantum number that depends on the particular FQH state.

For our investigations of the FQH at filling factor $\nu = 5/2$ we study a half-filled second LL, while the two lower-lying filled levels are considered as inert. For such a system the GS is at shift $S = 3$ (as obtained by exact diagonalizations¹³), the same value as for the MR state.³² We can insert N_{QH} $e/4$ -charged QHs into the system by adding $\frac{1}{2}N_{\text{QH}}$ supplementary flux quanta, i.e.,

$$N_\phi(N, N_{\text{QH}}) = 2N - 3 + N_{\text{QH}}/2. \quad (1)$$

In the spirit of our previous work⁶ we consider interactions of the form

$$V = (1 - x)V_C^{(1)} + xV_{3b}^{(0)} + V_{\text{pin}}(x), \quad (2)$$

where $0 \leq x < 1$, interpolating between the Coulomb interaction V_C and the three-body interaction V_{3b} , whose exact GS is the MR wave function,⁸

$$V_{3b} = \frac{A}{N^5} \sum_{i < j < k}^N S_{ijk} \{ \Delta_j \delta(i - j) \Delta_k^2 \delta(i - k) \}, \quad (3)$$

when projected onto the lowest (spin-polarized) LL. Here S_{ijk} expresses the full symmetrization over the permutations within the triplet (ijk) and $\delta(i - j)$ is the δ function in the separation of particles i and j . The constant A is chosen as in the earlier work,⁶ such that the gap for V_{3b} is approximately the same as that for V_C (note that V_{3b} induces an extensive energy for all GS at any $\nu > 1/2$).

The superscripts (0) and (1) in Eq. (2) indicate that the Coulomb energy $V_C^{(1)}$ is evaluated in the $n = 1$ LL, as required for the experimentally realized $\nu = 5/2$ state, while the three-body interaction $V_{3b}^{(0)}$ is evaluated in the $n = 0$ LL, since the MR state is the GS of V_{3b} in the lowest LL.

The pinning term $V_{\text{pin}}(x)$ localizes the QHs at fixed positions on the surface of the sphere. In our first calculations (in Secs. III and IV A) we consider a parametrization of the pinning potential of the form

$$V_{\text{pin}}(x) = (1 - x) \cdot [(1 - x)\mathcal{V}_{\text{pin}}^{(1)} + x\mathcal{V}_{\text{pin}}^{(0)}], \quad (4)$$

where the pinning function $\mathcal{V}_{\text{pin}}$ is still to be chosen, such that it localizes exactly an $e/4$ -charged QH. This $V_{\text{pin}}(x)$ couples to the electron density, interpolating as the electron interaction between LL $n = 1$ and $n = 0$ with weights $(1 - x)$ and x , respectively. The overall factor $(1 - x)$ makes the localization term vanish in the MR limit as $x \rightarrow 1$, just like the Coulomb term, both helping to separate the energies of states with charge $e/4$ QHs.

However, in the MR limit ($x \rightarrow 1$), the $e/2$ -charged “double QH” has a zero in the electron density and thus zero pinning energy, while the $e/4$ -charged QH shows a finite local minimum of the density and thus finite pinning energy. To separate $e/4$ -charged QHs requires either an additional repulsive interaction between QHs or a special pinning potential coupling to the monopole and quadrupole moment of the QHs (cf. problems discussed in Tóke *et al.*¹²). To avoid this difficulty, in this first series of calculations we never reach exactly the MR point, but approach it closely enough to identify the lowest-energy states. In Sec. IV B we will then use a different localization potential³³ to circumvent this problem and study the full AC up to the MR point.

All energies are measured in the usual units $e^2/\epsilon\ell_0$, where $\ell_0 = \sqrt{\hbar c/eB}$ is the magnetic length, and taking the energy of the lowest state as reference.

III. LOCALIZATION OF QUASIHOLE

We start by studying the QH localization in the pure Coulomb limit, taking for $\mathcal{V}_{\text{pin}}$ a δ -function pinning potential: Choosing a suitable pinning strength we see that it is actually possible to obtain separated $e/4$ -charged QHs, in contrast

FIG. 1. (Color online) Polar angle dependence of electron density and missing charge for $\nu = 5/2$ systems with two QHs localized at the poles of the Haldane sphere, for Coulomb interaction. In (a) the QHs are localized by a δ -function potential $\mathcal{V}_{\text{pin}}$ [setting $x = 0$ in Eq. (4)], for two different pinning strengths; in (b) by a smooth pinning potential, whose radial dependence is shown in (c). (d) Energy spectrum of a system with two QHs, without pinning, as function of the total angular momentum L , and comparison with the Coulomb energy for δ function and smooth pinning [cf. the text for more details; results are for $N = 20$, $N_\phi = 38$ in (a) and for $N = 18$, $N_\phi = 34$ in (b), (c) and (d)].

to the results of Tóke *et al.*¹² Figure 1(a) shows the polar angle dependence of the density (upper curves) and of the integrated missing charge (lower curves) for two different pinning strengths, for a system containing two QHs with localizing δ functions, one at the north and the other at the south pole of the sphere. We observe that the two QHs are localized at the poles. However they are large in extent and are accompanied by significant and slowly decaying density oscillations (cf. Nuebler *et al.*¹⁷). The missing charge $e/4$ is reached (for the stronger pinning potential) at $R \approx 4\ell_0$ and at the equator as required by symmetry.

As the system is compressible at the position of a QH, the shape of the pinning potential will influence its structure. We therefore study how the pinning potential can be tuned to minimize the size of the QH and to suppress the density oscillations around it. With the choice of shape, depicted in Fig. 1(c), we obtain an electron density without oscillations, as illustrated in Fig. 1(b): The smooth potential allows to properly separate the two QHs and to reduce their radius to a minimum of $\sim 3\ell_0$.

How is this possible? Figure 1(d) shows a comparison of the interaction energies, not including the energy due to the pinning forces, for δ -function pinning (upper line) and for smooth pinning (lower line). The circles depict the spectrum of a system with two free QHs, i.e., without pinning force, as a function of the total angular momentum L . We see that the use of a smooth pinning potential allows to obtain a smaller interaction energy, admixing only the lowest-lying L eigenstates, while the δ -function potential admixes also higher-lying states, giving rise to the large density oscillations. We also note that a “better” localization, with a pinning potential that further lowers the interaction energy, is unlikely to be possible, because a minimal number of low-energy L eigenstates is needed to obtain the desired QH localization.

We note that the energy of two-QHs systems is affected by how well the density oscillations fit with the separation distance of the QHs, such that the energy will show oscillations as function of the QH separation. The energy oscillations observed in the variational calculation of the Majorana fermion energy²⁹ may to some extent be caused by such commensuration effects. An optimized localization potential might help to reduce them, hopefully allowing a more accurate calculation of the coherence length associated with the Majorana fermion.

In the following, we use δ -function pinning potentials. We first study what happens if only a single δ function is inserted in a system of $N = 16$ electrons with two QHs, and whether they are still separated or if a single $e/2$ -charged “double quasi-hole” is formed. Figure 2(a) shows the energy spectrum as function of the angular momentum component L_z when two δ functions are inserted at the poles, for weak and strong pinning—cf. Fig. 1(a). In both cases the lowest-energy state has $L_z = 0$ and describes two separated QHs localized at the poles. Figure 2(b) shows the same, but for a single δ function at the south pole. The lowest-energy state has also $L_z = 0$, indicating that the QHs are still at maximal separation, which minimizes their Coulomb repulsion: The QH pinned at the south pole repels the other to the north pole of the sphere. For increasing pinning strength some states with $L_z \neq 0$ are reduced in energy. With strong enough pinning, a state with $L_z \neq 0$, in which QHs would cease to be maximally separated, may become the lowest-energy state. Of particular interest is the state at $L_z = N/2 = 8$, which corresponds to a “double QH” with charge $e/2$ localized at the south pole, but no charge deficiency at the north pole. At $\nu = 5/2$, for Coulomb interaction, this “double QH” state has quite a large interaction energy and, except perhaps in the limit of very strong pinning, is unlikely to become the GS.

FIG. 2. (Color online) (a) Energy spectrum of a two-QH system as a function of the angular momentum component L_z when δ functions are inserted at both poles (for two different pinning strengths, $N = 16$, $N_\phi = 30$); the lowest-energy state is the reference. (b) The same for a single δ function at the south pole.

IV. ADIABATIC CONTINUITY

A. Quasiholes localization with δ -function pinning

We now turn to the study of AC for systems containing localized QHs. We begin with two QHs, pinned at the poles by δ functions, and we vary the particle interaction as in Eq. (2) from $x = 0$ (the Coulomb case) to near $x = 1$ (the MR limit). Figure 3 shows the evolution of the excitation energies as function of the parameter x for $N = 12$ electrons and $N_\phi = 22$ flux units; other system sizes give similar results. The horizontal line at zero energy represents the GS energy: Clearly none of the lines for the excited states come close to it and there is no mixing between them. Thus there is no phase transition in going from the Coulomb case to the MR limit, exactly as observed for systems without QHs.⁶

Next we investigate a system of $N = 14$ electrons and $N_\phi = 27$ flux quanta that contains $N_{\text{QH}} = 4$ localized QHs.³⁴ This is particularly interesting because in the MR limit the GS is a degenerate doublet,³⁵ which is associated with a Majorana fermion²⁸ and with the non-Abelian braiding statistics of the QHs.^{26,27} For any four-QH configuration there are two linearly independent wave functions that describe it; braiding QHs around each other induces a linear transformation in the

FIG. 3. (Color online) Low-lying energy spectrum as function of the interaction parameter x for a system of $N = 12$ electrons with two QHs localized at the poles by δ -function pinning potentials; the lowest-energy state is the reference.

degenerate subspace of the doublet. Does this doublet survive in the Coulomb limit?

Figures 4(a) and 4(b) show results for four QHs localized at the corners of a square on a great circle and at the vertices of a tetrahedron, respectively. In both cases we see that the two lowest-energy states close to the MR limit, i.e., the GS at energy 0 and the first excited state, remain the lowest-energy states over the entire parameter interval, even at the Coulomb point $x = 0$. During the whole evolution there is neither crossing nor mixing with higher-lying states, indicating the absence of a phase transition. Note that in the MR case, at $x = 1$, all the considered states have zero energy, because the pinning potential $V_{\text{pin}}(x)$ vanishes at this point.

In the tetrahedral case, right in the Coulomb limit, a higher-energy state comes down in energy and nearly reaches the upper state of the doublet. But Fig. 4(d) shows that this is actually not a problem: The Coulomb spectrum is plotted as function of the strength of the pinning potential and it is evident that the near degeneracy is easily lifted by increasing the pinning strength. We also note that the two states of the doublet are not degenerate in the presence of a Coulomb interaction but we expect the splitting to vanish in the thermodynamic limit.²⁹

The curves denoted by “Pfaffian” in Figs. 4(a) and 4(b) show the variational results for the splitting of the doublet, computed using the states at $x = 0.95$ (compare with Baraban *et al.*²⁹): They overestimate the splitting in the Coulomb limit, in the square geometry by a factor of 2.5 and in the tetrahedral geometry by $\sim 70\%$.

Finally, Fig. 4(c) shows the evolution of the overlaps $\langle \psi_i(0) | \psi_i(x) \rangle$ of the two states $\psi_i(x)$ ($i = 0, 1$) of the doublet as the interaction varies from Coulomb V_C at $x = 0$ to the vicinity of the three-body limit V_{3b} at $x = 0.95$. We see that there is no sign of an abrupt drop of the overlaps which could signal a phase transition, and the values reached near the MR limit are reasonably large for a system of this size ($N = 14, N_\phi = 27$).

FIG. 4. (Color online) Low-lying energy spectrum as function of the interaction parameter x for systems with four QHs localized with δ -function pinning potentials: (a) At the corners of a square on a great circle and (b) at the vertices of a tetrahedron; the curves labeled Pfaffian give the variational estimate of the doublet splitting obtained using the GS doublet for the MR limit. (c) Overlaps of the two doublet states for Coulomb interaction with the corresponding states for interaction parameter x . (d) Low-lying energy spectrum for Coulomb interaction, as function of the pinning strength, for four QHs in tetrahedral position. (The lowest-energy state is the reference; $N = 14$, $N_\phi = 27$.)

All these results provide evidence in favor of adiabatic continuity from the MR to the Coulomb limit for the two lowest-energy states in systems with four localized QHs, suggesting that the non-Abelian doublet, and the associated Majorana fermion, can be expected in the limit of pure Coulomb interaction.

B. Quasiholes localization with “STM tip” pinning

In the AC investigations described so far, the pinning potential $V_{\text{pin}}(x)$ [in Eq. (4)] was not able to localize QHs in the MR limit (as discussed in Sec. II), but we approached this limit near enough to recognize the lowest-lying states. One can pose the question if the permanent presence of the Coulomb interaction (that vanish as $1 - x$ in the MR limit $x \rightarrow 1$) could influence our results, selecting exactly the “right states” from the zero energy states at the MR point. To strengthen our evidences and to show a complete AC between the Coulomb and MR limit, we repeat here the calculations for systems containing four QHs, using a different pinning potential $V_{\text{pin}}(x)$ that allows to reach both limits:

$$V_{\text{pin}}(x) = (1 - x)V_q^{(1)} + xV_q^{(0)}, \quad (5)$$

where V_q , interpolated between the $n = 1$ and the $n = 0$ LL as the electron interaction, is the Coulomb potential of a pointlike object, e.g., a STM tip, with a (negative) charge q , positioned on the surface of the FQH sample.³³ This potential repels the electrons from its center and, if the charge q is chosen in an appropriate way, namely, in an interval near $-e/4$, it can only localize $e/4$ -charged QHs, thus avoiding the problem of the localization of $e/2$ -charged “double QHs” in the MR limit. This idea was first used by Prodan and Haldane³⁶ in their investigation of the (non-Abelian) braiding properties of MR QHs, although projecting the pinning potential on the zero energy space of their Hamiltonian.

Figure 5 shows the results for the low-lying energy spectrum, as the interaction is varied from the Coulomb ($x = 0$) to the MR limit ($x = 1$), for systems with four QHs localized at the corners of a tetrahedron by four such pinning potentials. The results in Fig. 5(a) are for $N = 12$ electrons and a localizing charge $q = -e/4$, while in Fig. 5(b) for $N = 14$

electrons³⁴ and $q = -0.19e$: These charges were chosen in order to obtain that the two states of the MR doublet are exactly the lowest-lying states in the MR limit.

FIG. 5. (Color online) Low-lying energy spectrum as function of the interaction parameter x for systems with four QHs localized at the vertices of a tetrahedron using the pinning potential in Eq. (5): (a) $N = 12$, $N_\phi = 23$, $q = -e/4$, (b) $N = 14$, $N_\phi = 27$, $q = -0.19e$ (the lowest-energy state in the MR limit is the reference).

In both cases we see that (once q is fixed) the two lowest-energy states remain the lowest over the whole parameter interval (for $N = 12$ with a level crossing). The third energy level, corresponding to an excitation in the pinning potential, is exactly threefold degenerate (for symmetry reasons) and can thus be easily distinguished from the two lowest states forming the MR doublet.

We note that this doublet has a finite energy splitting, not only for pure Coulomb interaction, but also in the MR limit: This is caused by the pinning potential (5) that mixes in states with nonzero energy. We also find that the energy separation between the upper state of the doublet and the third (threefold degenerate) state can be small, particularly in the MR limit, and depends on the charge q used in the pinning potential. For example, for $N = 14$ electrons, using a localizing charge $q = -e/4$, the pinning potential is too strong and at the MR point the threefold degenerate state is slightly lower in energy than the upper state of the doublet; to obtain the right order we chose a weaker potential, with $q = -0.19e$.

The AC between the Coulomb and the MR limit for the lowest-energy doublet shown in Fig. 5 thus supports the evidences of the previous section in favor of a “survival” of the Majorana fermion till the Coulomb point.

In Fig. 6 we also show the particle density on the surface of the sphere for the two states of the MR doublet in the two limiting cases: (a), (b) for the MR and (d), (e) for the Coulomb limit. The two states of the doublet are indeed very similar, as also shown in (c) and (f), where the density differences are plotted. Note that the density differences in the MR case are approximately twice as large as the differences in the Coulomb limit.

V. QUASIHOLE BRAIDING AND FUSION

In this last section we show some preliminary results of QH braiding and fusion in systems containing four localized QHs. We perform this investigation for electron interactions near to the Coulomb limit, but slightly increasing v_1 , the Haldane pseudopotential³⁰ that describes the interaction between two particles having a relative angular momentum $l\hbar$. In our previous work⁶ we showed that, under such interaction modifications, the system remains in the MR phase, even improving the overlap between the exact GS and the Pfaffian wave function and simultaneously with an enhancement of the gap. For the QH localization we use the pinning potential introduced in the previous section [Eq. (5)], setting $x = 0$, that is evaluating it fully in the $n = 1$ LL, and choosing a suitable value for the localizing charge q , as explained below.

Investigating QH braiding in systems with four QHs, the non-Abelian nature of the MR doublet states should become manifest: Interchanging the positions of two QHs by stepwise changing the location of their pinning potentials (and keeping the other two QHs fixed), the two states of the MR doublet should transform into each other, in particular, switching their position in the energy spectrum. For this to happen, during the QH's interchange process, an odd number of level crossings between them is needed. In the following calculations, we engineer an exact energy degeneracy for the two states of the doublet at the midpoint of the QH's interchange process, by choosing suited values for the localizing charge q and the first Haldane pseudopotential v_1 . Then, performing the interchange, we investigate if this is indeed a crossing point and whether it is the only one or if others arise.

FIG. 6. (Color online) Particle density on the sphere surface for the two states of the MR doublet: (a), (b) in the MR limit and (d), (e) in the Coulomb case; (c) and (f) show the density difference between the two states ($N = 14$, $N_\phi = 27$).

We chose as a starting (and ending) configuration four QHs at the corners of a rectangle: two of them on the upper half-sphere, at the same polar angle $\theta_{1,2} \equiv \theta_1 = \theta_2 = 54.736^\circ$ and at the opposite azimuthal angles $\varphi_1 = 90^\circ$, $\varphi_2 = 270^\circ$; the other two on the lower half-sphere at the polar angle $\theta_{3,4} \equiv \theta_3 = \theta_4 = 125.264^\circ$ and azimuthal angles $\varphi_3 = 90^\circ$ and $\varphi_4 = 270^\circ$. We keep QHs 3 and 4 fixed at their locations and, by stepwise changing the positions of the pinning potentials, we rotate both QH 1 and 2 around the vertical axis through the poles,

$$\varphi_1(\varphi) = 90^\circ + \varphi, \quad \varphi_2(\varphi) = (270^\circ + \varphi) \bmod 360^\circ, \quad (6)$$

keeping the polar angle $\theta_{1,2} = 54.736^\circ$ constant, until they exchange their original positions: $0^\circ \leq \varphi \leq 180^\circ$. This is a 180° -periodic process; we call φ the “braiding angle.”

The polar angles $\theta_{1,2}$ and $\theta_{3,4}$ are chosen in such a way that at the midpoint of the rotation process, that is at braiding angle $\varphi = 90^\circ$, the four QHs are in a tetrahedral configuration. The localizing charge q and Haldane pseudopotential v_1 (for the whole braiding process) were previously chosen, such that at this point the two states of the MR doublet are degenerate in energy: $q = -0.1715e$, $v_1 = 1.05v_1^{\text{Coul}}$ for $N = 12$ and $q = -0.19e$, $v_1 = 1.075v_1^{\text{Coul}}$ for $N = 14$ electrons, where v_1^{Coul} is the Coulomb value of the pseudopotential v_1 . These are “reasonable” values for q and v_1 : The localized QHs have charge $e/4$ and the electron interaction is modified such that the system is still in the MR phase.

Figure 7 shows the evolution of the low-lying energy spectrum as function of the braiding angle φ for (a) $N = 12$ and (b) $N = 14$ electrons. In both cases we observe that the two states of the doublet remain the lowest-energy states during the whole braiding process, never mixing with higher-lying states or with each other (except at the degeneracy point). Indeed, the overlaps between the corresponding states of the doublet for successive braiding steps are consistently high: in the range 0.94 – 0.97 for $N = 12$ (5° steps) and 0.97 – 0.99 for $N = 14$ (3° steps).

From the plots it is evident that the degeneracy point at $\varphi = 90^\circ$ is indeed also a crossing point for the two doublet states: If one follows the lines with a continuous derivative, at the degeneracy point one goes from the lower to the upper state of the doublet (and vice versa). No other crossing or mixing point arises and thus the QH braiding causes an interchange in the MR doublet states: If one starts at $\varphi = 0^\circ$ in the lower-energy state, one ends at $\varphi = 180^\circ$ in the upper-energy state, as expected for non-Abelian braiding statistics. Two such braidings are needed to come back to the initial state.

We note that we can get an exact degeneracy of the MR doublet only when the four QHs are in the tetrahedral configuration. Slightly modifying the geometry of the configuration, a small gap opens; however, if the speed of the braiding is sufficiently high, the states of the MR doublet would still interchange at this point. This situation is analogous to that studied by Thouless and Gefen,³⁷ concerning the crossing between the lowest-lying states as function of the magnetic flux, for the quantum Hall effect at fractional fillings, showing that these crossings are essential in order to get a fractional charge. On the other hand, if the QHs are moved in a strictly adiabatic way, the system always remains in

FIG. 7. (Color online) Braiding of two QHs (in the presence of two other fixed QHs): low-lying energy spectrum as function of the braiding angle φ , defined through Eq. (6). The electron interaction is near to the Coulomb limit (with a slightly increased Haldane pseudopotential v_1) and the QHs are localized by the pinning potential of Eq. (5), setting $x = 0$. (a) $N = 12$, $q = -0.1715e$, $v_1 = 1.05v_1^{\text{Coul}}$, (b) $N = 14$, $q = -0.19e$, $v_1 = 1.075v_1^{\text{Coul}}$, where v_1^{Coul} is the Coulomb value of the first Haldane pseudopotential.

the lowest-energy state, without crossing. We also wish to emphasize that the doublet degeneracy is very different from the threefold degeneracy of the first excited state: The former is obtained only by fine tuning of the parameters q and v_1 , the latter is a purely geometric degeneracy, resulting from the symmetric QH configuration and independent from interaction and localization parameters.

The fact that the MR doublet states are not degenerate (except by tuning at $\varphi = 90^\circ$) in our small system diagonalizations helps us to follow their evolution during the braiding (through the evolution of the energy levels) and thus to recognize the crossing point, which leads to the interchange of states. However, we expect that in the thermodynamic limit the two states become exactly degenerate during the whole braiding process, when the QHs are sufficiently far apart from each other: On one side this would make the fine tuning of the parameters q and v_1 unnecessary, but on the other side it would make it impossible, using our present method, to follow the evolution of the states under braiding (however, it is still possible with a study of the monodromy matrix, as was done

FIG. 8. (Color online) Fusion of two QHs at the north pole (in the presence of two other fixed QHs): low-lying energy spectrum for $N = 14$ electrons as function of $\theta_{1,2} \equiv \theta_1 = \theta_2$, the polar angle of both the moving QH 1 and 2. The electron interaction is near to the Coulomb limit (with a slightly increased Haldane pseudopotential $v_1 = 1.075v_1^{\text{Coul}}$) and the QHs are localized by the pinning potential of Eq. (5), setting $x = 0$ and $q = -0.19e$ (v_1^{Coul} is the Coulomb value of the first Haldane pseudopotential).

by Prodan and Haldane³⁶); nevertheless, the interchange of the MR doublet states should survive also in this limit.

Unfortunately we can get results for QH braiding only for small system sizes³⁴ and from them we cannot extract useful information about the dependence of the energy splitting when approaching the thermodynamic limit. The average splitting of the MR doublet for $N = 14$ is slightly smaller than for $N = 12$, but this is mainly caused by the different shape of the energy oscillations as function of the braiding angle φ in the two cases: For $N = 14$ the oscillations of the MR doublet states are almost “in phase,” while for $N = 12$ they are “out of phase,” thus giving a larger average splitting.

Finally we investigate the fusion of two QHs by bringing the pinning potentials that localize them close to each other, in the presence of two others fixed QHs. We start from the tetrahedral configuration described above and we let fuse the two QHs in the upper half-sphere, by taking them to the north pole, that is, by shrinking the polar angle $\theta_{1,2} \equiv \theta_1 = \theta_2$ from 54.736° to 5° at fixed azimuthal angles. Figure 8 shows the low-lying energy spectrum as function of the polar angle $\theta_{1,2}$ during this process, for $N = 14$ electrons. We see that, after that the degeneracy is lifted, the evolution of the two lowest states is similar, with a relatively constant splitting between them. This splitting (or at least a part of it) could come from the intrinsic splitting of the MR doublet as two QHs are taken together (its order of magnitude is the same as that obtained by Baraban *et al.*²⁹). However, at approximately $\theta_{1,2} = 37^\circ$, the upper state of the MR doublet comes very near to the next higher-lying state and strongly mixes with it³⁸ (but without crossing), possibly losing a part of its character. Thus it is not fully clear what happens at the MR doublet, and further investigations are needed.

VI. CONCLUSION

Our results provide evidence in favor of adiabatic continuity from the Moore-Read to the Coulomb limit for the two lowest-energy states in systems with four localized QHs and thus the non-Abelian doublet and associated Majorana fermion can be expected in the limit of pure Coulomb interaction. Forces breaking particle-hole symmetry as V_{3b} are actually present due to LL mixing^{39,40} and will favor either the Pfaffian⁷ or the anti-Pfaffian.^{41,42}

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