Signal-to-noise ratio in x ray dark-field imaging using a grating interferometer

Michael Chabior,^{1,a)} Tilman Donath,^{2,3} Christian David,² Manfred Schuster,^{1,b)} Christian Schroer,⁴ and Franz Pfeiffer⁵

¹Siemens AG, Corporate Technology, München 80200, Germany

²Paul Scherrer Institut, 5232 Villigen PSI, Switzerland

³Dectris Ltd., Neuenhoferstrasse 107, Baden 5400, Switzerland

⁴Technische Universität Dresden, Dresden 01062, Germany

⁵Technische Universität München, Garching 85748, Germany

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In this work, we report an analytical and experimental investigation of the signal-to-noise ratio for a recently developed method called x ray dark-field imaging. Our approach is based on the propagation of signal and noise through the reconstruction algorithm. We find that the statistical nature of the dark-field images can be understood by a Rician distribution. The analysis shows that, for high flux, the noise in the dark-field images is proportional to the noise in the raw data. In the limit of low flux and, thus, low signal-to-noise ratio, the dark-field signal exhibits a breakdown of the signal transmission, which can be described by an asymptotic behavior of the underlying noise distribution. In this limit, the dark-field signal is no longer connected to the coherence degradation, but rather to the attenuation in the sample. The model is verified in exemplary test measurements using a compact laboratory setup with a polychromatic source and a photon counting detector. © 2011 American Institute of Physics. [doi:10.1063/1.3630051]

I. INTRODUCTION

Since their discovery more than 100 years ago, x rays have been widely utilized for imaging applications in medical diagnostics and nondestructive testing. Recent developments in x ray phase-contrast imaging^{1–10} have opened the way for possible applications of phase-enhanced contrast modalities in biomedical research and clinical diagnostics.¹¹ Various methods are currently evaluated, and the grating-based differential phase-contrast technique appears as one of the most promising new approaches. It offers the advantage that it is not only capable of delivering phase-contrast images, but also conventional attenuation-contrast images in the same measurement. Moreover, the method is compatible with spatially incoherent, high-powered, polychromatic laboratory x ray sources,^{12,13} so that clinical applications remain feasible. Beyond the attenuation and phase-contrast channels, the method is capable of creating a contrast which is related to the internal structure of the transmitted object:¹⁴ diffuse small-angle x ray scattering within the sample leads to a coherence degradation of the wavefront, which can be used for image formation. This paper aims at understanding the statistical properties of the dark-field-contrast channel by analyzing the propagation of noise in the raw data through the reconstruction algorithms.

First, the basic principles of grating-based differential phase-contrast imaging will be revised. Formulas for reconstructing the dark-field signal are given, together with a discussion of noise propagation. The statistical behavior of the noise present in dark-field images will then be described as a

transition between the high and low signal-to-noise ratio (SNR) limit. A verification of the developed formulas in exemplary measurements concludes the investigation.

II. GRATING-BASED X RAY DARK-FIELD IMAGING

In grating-based differential phase-contrast imaging, the contrast is generated using a grating-interferometer, as depicted in Fig. 1: a central beam splitter grating G_1 generates an interference pattern downstream of the grating. A sample placed in the beam path will create distortions in this interference pattern, which open the possibility of reconstructing the phase shift caused by the sample. As in most experimental cases, the interference pattern is too small to be resolved directly by common x ray detectors; a second analyzer grating G_2 is used to analyze the fringes in a Moiré approach (see Ref. 10 for a deeper explanation of the technique). A third attenuation grating G_0 is used to split the broad focal spot of a high-powered laboratory source into an array of line sources, each meeting the lateral coherence requirements of the technique. The period of the grating is adjusted to the period of the interference pattern, so that the images created by each line source overlap uniformly at the plane of detection.¹² A lateral phase-stepping scan of one of the gratings creates an oscillating intensity signal S_n in each detector pixel as a function of the grating position. In most experimental situations, the intensity oscillation S_n in each pixel is approximately harmonic and can be described by a Fourier expansion, which is truncated after the second term. If a number of $N \geq 3$ equidistant steps over an integer number of periods of the interference pattern are measured in the phase-stepping scan, i.e., $n = 0, \dots, N - 1$ with $S_0 = S_N$, the oscillation can be written as

^{a)}Electronic mail: michael.chabior.ext@siemens.com.

^{b)}Electronic mail: schuster.manfred@siemens.com.

FIG. 1. Schematic measurement setup and sample arrangement. G_0 is the source grating, G_1 the beam splitter grating used for the phase stepping, and G_2 the analyzer grating. The samples are a square $10 \times 10 \text{ mm}^2$ PMMA bar aligned along the diagonal and a square $5 \times 5 \text{ mm}^2$ Al bar.

$$S_n = a_0 + a_1 \cdot \cos\left(\frac{2\pi}{N}n - \varphi\right). \quad (1)$$

We will name the parameter a_0 the mean, a_1 the amplitude, and φ the phase of the phase-stepping oscillation. The phase φ is related to the differential phase shift caused by the sample, which will not be followed further in this paper (see Refs. 15 and 16 for a discussion of the noise properties of this signal). The measured intensity oscillation S_n can be analyzed using the inverse discrete Fourier transform. The k -th Fourier coefficient c_k can be written as

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} S_n \cdot e^{\frac{2\pi i}{N}nk}. \quad (2)$$

As we assume that the intensity oscillation is harmonic, only the zeroth and first coefficient carry information. The offset a_0 of the oscillation can be reconstructed from the zeroth Fourier coefficient, which is equal to the mean value of all projections S_n ,

$$a_0 = c_0 = \frac{1}{N} \sum_{n=0}^{N-1} S_n. \quad (3)$$

The oscillation amplitude a_1 is identical to twice the magnitude of the first complex Fourier coefficient c_1 ,

$$a_1 = 2 \cdot |c_1| = 2\sqrt{\text{Re}^2(c_1) + \text{Im}^2(c_1)}. \quad (4)$$

The factor of two results from the equal contribution of negative frequency components. For an evaluation of the attenuation and coherence degradation caused by the measured sample, a reference scan has to be performed. If we denote the reference scan with the index r and the sample scan with the index s , the transmission T through the sample is calculated from the ratio of the mean values of the phase-stepping oscillation

$$T = \frac{a_{0,s}}{a_{0,r}} = \exp\left(-\int \mu \cdot dz\right), \quad (5)$$

where $\mu(E, x, y, z)$ is the linear attenuation coefficient of the sample material. A similar definition can be given for the decrease of the amplitude, which will be named A in the following:

$$A = \frac{a_{1,s}}{a_{1,r}}. \quad (6)$$

The oscillation amplitude a_1 depends mainly on the coherence of the utilized beam, which can be quantified by the known concept of visibility. The interferometric visibility ν can, in general, be calculated from the maximum and minimum intensity $I_{\max}$ and $I_{\min}$ of the oscillation by $\nu = (I_{\max} - I_{\min}) / (I_{\max} + I_{\min})$. For a harmonic oscillation pattern, the visibility is identical to the oscillation amplitude a_1 , normalized by the mean intensity a_0

$$\nu = \frac{a_1}{a_0}. \quad (7)$$

By quantifying the loss of visibility and, thus, the degradation of coherence, one obtains a contrast, which is denoted as dark-field contrast

$$D = \frac{\nu_s}{\nu_r} = \frac{A}{T}. \quad (8)$$

Note that the dark-field signal D is identical to the loss of oscillation amplitude A relative to the transmission T in the sample. The loss of coherence can be related to phase fluctuations of the x ray wavefront caused by small-angle x ray scattering at micron-sized structures within the sample.^{14,17–19} Thus, a contrast similar to dark-field imaging in light microscopy can be achieved in the x ray regime.

III. NOISE PROPAGATION AND NOISE DISTRIBUTIONS

Measured data are always subject to signal fluctuations of various origins, which are referred to as noise. The noise present in the recorded raw images $S_n(x, y)$ will propagate through the Fourier transform algorithms in Eqs. (2)–(4) and influence the quality of the images derived from the raw data. To simplify the analysis of the noise properties of our images, in the following, we will assume that the only source of noise is the Poissonian noise from photon-counting statistics, ignoring all other sources of noise as positioning jitter, alignment instabilities, electronic noise, etc. The signal strength per pixel S_n is thus equal to the number of detected photons per pixel and phase-step, with variance $V(S_n) = S_n$, following Poissonian statistics. As the Gaussian distribution is a good approximation to the Poissonian distribution for all count rates of interest in this paper, we will assume that the raw data S_n are corrupted by an additive zero-mean Gaussian white noise with standard deviation $\sqrt{V(S_n)}$. We further assume an equidistant phase-stepping over an integer number of periods of the interference pattern. Under these assumptions, the expectation value $E(a_0)$ of the mean of the phase-stepping oscillation is

$$E(a_0) = a_0, \quad (9)$$

while the variance $V(a_0)$ of the reconstructed offset a_0 is

$$V(a_0) = \frac{a_0}{N}. \quad (10)$$

We will use the normalized standard deviation $\tilde{\sigma} := \sqrt{V(a_0)}/E(a_0)$ to quantify noise and for comparison with other x ray imaging modalities, like classical x ray radiography. Note that $1/\tilde{\sigma}$ is equal to the signal-to-noise ratio in the attenuation-contrast channel. Using Eqs. (9) and (10), $\tilde{\sigma}$ can be rewritten as

$$\tilde{\sigma} = \frac{1}{\sqrt{a_0 \cdot N}}. \quad (11)$$

Assuming a minimum of $N = 3$ phase steps and a minimum number of $a_0 = 10$ photons per step to be able to apply the Gaussian approximation to the Poissonian distribution, the results are expected to be valid for $\tilde{\sigma} < 0.2$.

As the dark-field signal is calculated by a relation of the mean a_0 and the amplitude a_1 , we need to have a closer look at $a_1 = 2\sqrt{\text{Re}^2(c_1) + \text{Im}^2(c_1)}$. As the Fourier transform is a linear function, the noise properties of the complex Fourier coefficients equal the noise properties of the raw data. Nevertheless, taking the absolute value of that complex number introduces a nonlinearity into the noise propagation. If the real and imaginary part of a complex number are corrupted by a zero mean Gaussian noise with variance $V(\text{Re}(c_1)) = V(\text{Im}(c_1)) = \sigma_c^2$, it is well known in signal processing that consecutive measurements of the amplitude of that complex number follow a Rician distribution.^{20–22} The probability density function (PDF) of a Rician process can be described by

$$f(x, a_1, \sigma_c) = \frac{x}{4\sigma_c^2} \cdot \exp\left(-\frac{(x^2 + a_1^2)}{8\sigma_c^2}\right) \cdot I_0\left(\frac{x \cdot a_1}{4\sigma_c^2}\right), \quad (12)$$

where x is the random variable or reconstructed amplitude, a_1 and σ_c the parameters of the distribution, and I_0 the modified Bessel function of the first kind of order zero. The expectation value of a Rician PDF is equal to

$$E(x) = \sigma_c \sqrt{2\pi} \cdot L_{\frac{1}{2}}\left(-\frac{a_1^2}{8\sigma_c^2}\right), \quad (13)$$

with $L_{\frac{1}{2}}(x)$ being a Laguerre polynomial $L_m(x)$ with $m = 1/2$, while the variance is

$$V(x) = a_1^2 + 8\sigma_c^2 - E(x)^2. \quad (14)$$

Note that Eqs. (12)–(14) are different from classical definitions of a Rician PDF, due to the fact that a_1 equals twice the magnitude of a complex number. The Rician PDF for four SNR values is shown in Fig. 2. While the distribution is well described by a Gaussian PDF for high SNR ($a_1/\sigma_c > 3$), it approaches a Rayleigh distribution (i.e. a χ distribution with two degrees of freedom) when the SNR decreases. Instead of describing the statistics of our reconstructed parameters in terms of the Rician distribution, we chose the more insightful approach to model the statistical behavior as a transition between the two characteristic noise distributions in the high

FIG. 2. Plot of a Rician distributed amplitude for constant $a_1 = 100$ and four different values of σ_c . As the distribution broadens with increasing noise level, it becomes skew and its mean shifts toward higher values.

and low SNR limit, namely a Gaussian distribution and a Rayleigh distribution, respectively. In the following, we will denote the oscillation amplitude by a_1 and the reconstructed amplitude in the high SNR limit by $\hat{a}_1$ and in the low SNR limit by $\check{a}_1$. In the high SNR limit, the amplitude distribution will thus approximately follow a Gaussian PDF

$$f(\hat{a}_1, \sigma_c) = \frac{1}{\sqrt{8\pi\sigma_c^2}} \exp\left(-\frac{(\hat{a}_1 - a_1)^2}{8\sigma_c^2}\right), \quad (15)$$

with the expectation value

$$E(\hat{a}_1) = a_1 \quad (16)$$

and the variance

$$V(\hat{a}_1) = 4\sigma_c^2. \quad (17)$$

For decreasing SNR, the distribution of the reconstructed amplitudes $\check{a}_1$ will tend towards a Rayleigh distribution. Its PDF can be obtained from Eq. (12) in the limit of $a_1 \rightarrow 0$, leading to

$$f(\check{a}_1, \sigma_c) = \frac{\check{a}_1}{4\sigma_c^2} \exp\left(-\frac{\check{a}_1^2}{8\sigma_c^2}\right). \quad (18)$$

Using the properties $a_1 \rightarrow 0$ and $L_{\frac{1}{2}}(0) = 1$ in Eqs. (13) and (14), its expectation value can be given by

$$E(\check{a}_1) = \sigma_c \cdot \sqrt{2\pi} \quad (19)$$

and its variance by

$$V(\check{a}_1) = \sigma_c^2(8 - 2\pi). \quad (20)$$

The variance σ_c^2 of the noise PDF of the real and imaginary part of the complex Fourier coefficients is determined by the noise in the raw images and can thus be related to the noise in the attenuation-contrast channel by

$$\sigma_c = \sqrt{\frac{V(a_0)}{2}} = \tilde{\sigma} \frac{a_0}{\sqrt{2}}. \quad (21)$$

Combining Eqs. (16), (17), and (21), the relative reconstruction error of the amplitude in the high SNR limit as a function of $\tilde{\sigma}$ is

$$\frac{\sqrt{V(\hat{a}_1)}}{E(\hat{a}_1)} = \tilde{\sigma} \cdot \frac{\sqrt{2}}{\nu}. \quad (22)$$

In the low SNR limit, the relative reconstruction error can be obtained by combining Eqs. (19) and (20), leading to

$$\frac{\sqrt{V(\tilde{a}_1)}}{E(\tilde{a}_1)} = \sqrt{\frac{4 - \pi}{\pi}}. \quad (23)$$

Using these considerations on the reconstruction of the amplitude, the reconstruction of the visibility can be analyzed. We will not derive the PDF of the underlying ratio distribution of the visibility, but instead describe its expectation value and variance by the PDFs of the amplitude and mean. If the function $\nu(a_0, a_1) = a_1/a_0$ is expanded into a power series around $E(a_0)$ and $E(a_1)$, it is straightforward to calculate that, in the absence of correlation between a_0 and a_1 , the expectation value of the reconstructed visibility is given by

$$E(\nu) = \frac{E(a_1)}{E(a_0)} \sum_{k=0}^{\infty} \frac{M_k(a_0)}{E^k(a_0)}, \quad (24)$$

where $M_k(a_0)$ is the k -th central moment of the distribution of a_0 . Thus, we may approximate the expectation value of the reconstructed visibility by $E(\nu) \approx E(a_1)/E(a_0)$ up to second order, as long as $V(a_0)/E^2(a_0) = \tilde{\sigma}^2 \ll 1$. Under these conditions, the variance of the reconstructed visibility $V(\nu)$ is given by

$$V(\nu) \approx \frac{E^2(a_1)}{E^2(a_0)} \cdot \left(\frac{V(a_1)}{E^2(a_1)} + \frac{V(a_0)}{E^2(a_0)} \right). \quad (25)$$

These approximations will be implicitly used in the following, greatly simplifying the analysis.

The statistical properties of the amplitude distribution make it necessary to distinguish between a high and low SNR limit for the visibility as well, which will be denoted by $\hat{\nu}$ and $\tilde{\nu}$, respectively. While for high SNR, the expectation value is $E(\hat{\nu}) \approx E(\hat{a}_1)/E(a_0) = \nu$, by combining Eqs. (10) and (22), the relative error of the reconstructed visibility can be calculated by

$$\frac{\sqrt{V(\hat{\nu})}}{E(\hat{\nu})} = \tilde{\sigma} \cdot \sqrt{\frac{2}{\nu^2} + 1}. \quad (26)$$

Eq. (26) holds for high SNR, while its statistical behavior changes as the flux or exposure time decreases and the noise in the images increases. In the low SNR limit, the PDF of the reconstructed visibility will be that of a ratio of a Rayleigh and Gaussian distribution. Using again $E(\tilde{\nu}) \approx E(\tilde{a}_1)/E(a_0)$ in combination with the expectation value of the amplitude $E(\tilde{a}_1)$, in the limit of low SNR described by Eq. (19), we arrive at

$$E(\tilde{\nu}) = \tilde{\sigma} \sqrt{\pi} \quad (27)$$

for the expectation value and

$$V(\tilde{\nu}) = \tilde{\sigma}^2 (4 - \pi + \pi \cdot \tilde{\sigma}^2) \quad (28)$$

for its variance. Thus we obtain

$$\frac{\sqrt{V(\tilde{\nu})}}{E(\tilde{\nu})} = \sqrt{\frac{4 - \pi}{\pi} + \tilde{\sigma}^2}. \quad (29)$$

The visibility reconstructed from noisy raw data S_n in the limit of high and low SNR exhibits interesting properties: the visibility reconstruction is dominated by the statistical properties of the amplitude distribution. If the SNR is lowered in a measurement, the noise characteristics in the measured visibilities will show a transition from Gaussian distributed noise, as described by Eq. (26), to Rayleigh distributed noise, as described by Eq. (29). For high SNR, the relative errors of the reconstructed amplitude $\sqrt{V(\hat{a}_1)}/E(\hat{a}_1)$ and the visibility $\sqrt{V(\hat{\nu})}/E(\hat{\nu})$ in Eqs. (22) and (26) rise linearly with the noise level $\tilde{\sigma}$. Additionally, the errors of the reconstructed amplitude and visibility rise with decreasing visibility, in opposition to the attenuation contrast, which is independent of the oscillation visibility. In the limit of low SNR, Eq. (27) predicts that the expectation value of the reconstructed visibility is not constant, but instead a function of the noise level, eventually becoming larger than the visibility in the reference image. As the visibility is the carrier of information in a dark-field measurement, it is obvious that this corresponds to a loss of information through the application of the reconstruction algorithms. We thus can conclude that, by the transition in the statistical behavior of the underlying parameters, a lower bound for the necessary photon counts, or dose, is required to obtain reasonable dark-field images.

IV. CALCULATION USING A NOISY REFERENCE IMAGE

In most experimental situations, the reference images, which are needed for a proper evaluation of the image data, can be acquired with arbitrary accuracy and thus can be considered approximately noise free. The formulas presented in section III can then be used without further alteration. If, however, the reference images are assumed to be noisy as well, further analysis has to be performed. For the sake of simplicity, we will assume that the noise level in the reference images is equal to the noise level in the sample images. From Eq. (5), the expectation value of the reconstructed transmission equals $E(T) = T$, while the relative reconstruction error is

$$\frac{\sqrt{V(T)}}{E(T)} = \tilde{\sigma} \cdot \sqrt{1 + \frac{1}{T}}. \quad (30)$$

For high SNR, the expectation value of the reconstructed amplitude decrease is $E(\hat{A}) \approx E(\hat{a}_{1,s})/E(\hat{a}_{1,r}) = A$, while the relative reconstruction error reads

$$\frac{\sqrt{V(\hat{A})}}{E(\hat{A})} = \tilde{\sigma} \cdot \sqrt{\frac{2}{\nu_r^2} \left(1 + \frac{1}{T \cdot D^2} \right)}, \quad (31)$$

with ν_r denoting the visibility in the reference image. Similarly, the expectation value of the dark-field signal is $E(D) = E(v)/E(v) = D$. Using Eqs. (39) and (31), it is possible to calculate the relative reconstruction error of the dark-field signal by

$$\frac{\sqrt{V(\hat{D})}}{E(\hat{D})} = \tilde{\sigma} \cdot \sqrt{\frac{2}{\nu_r^2} \left(1 + \frac{1}{T \cdot D^2}\right) + \left(1 + \frac{1}{T}\right)}. \quad (32)$$

By comparing Eqs. (30) and (32), one can conclude that the noise in the dark-field contrast images will always be higher than in the attenuation-contrast images.

Eq. (32) only holds for high SNR, while in the limit of low SNR, the transition in statistical behavior dictates other relationships. Taking into account the results of section III, from Eq. (23) it immediately follows that the expectation value of the amplitude decrease $E(\check{A}) \approx E(\check{a}_{1,s})/E(\check{a}_{1,r})$ can be written in the low SNR limit as

$$E(\check{A}) = \sqrt{T} \quad (33)$$

and its relative reconstruction error can be calculated to be

$$\frac{\sqrt{V(\check{A})}}{E(\check{A})} = \sqrt{2 \cdot \frac{4 - \pi}{\pi}}. \quad (34)$$

If we assume $E(\check{D}) \approx E(\check{A})/E(T)$, we can finally write the dark-field signal in the low SNR limit using Eq. (33) as

$$E(\check{D}) = \frac{1}{\sqrt{T}}, \quad (35)$$

with a relative reconstruction error of

$$\frac{\sqrt{V(\check{D})}}{E(\check{D})} = \sqrt{2 \cdot \frac{4 - \pi}{\pi} + \tilde{\sigma}^2 \left(1 + \frac{1}{T}\right)}. \quad (36)$$

In the limit of low SNR, the dark-field signal will therefore not consist of plain noise, but instead will converge toward an image which is proportional to the inverse square root of the transmission image. This dependence can be understood due to the fact that, for low SNR, the visibility is a function of the noise level and thus inherently of the attenuation in the sample after Eq. (27).

With these analytical results, which are obtained by applying Rician statistics to the image reconstruction algorithm, we can resume to verify the obtained formulas in an exemplary measurement.

V. EXPERIMENTS

To prove the correctness of the theoretical approach and the resulting formulas, a series of measurements using some well-defined test samples were performed as a function of exposure time and, thus, image noise. The measurements were done at a laboratory setup at the Paul Scherrer Institut in Villigen, Switzerland. As an x ray source, a Seifert DX-W8x0.4-L x ray tube was used, operated at an anode voltage of 40 kV and an anode current of 25 mA. No filtration was

applied. As detector, a Dectris Pilatus 100 K was used. Detection is achieved by photon conversion in a reverse-biased Si diode array with 320 μm sensor thickness. The sensitive area consisted of a 487×195 pixel array with a pixel size of $172 \times 172 \mu\text{m}^2$, thus offering a total field of view of $83.8 \times 33.5 \text{ mm}^2$. The single photon counting detector could handle count rates of up to 2×10^6 photons per second, with a dynamic range of 20 bit. All gratings were manufactured at the Paul Scherrer Institut.²³ The phase grating G_1 had Si bars of 35 μm height and a period of 3.5 μm , exhibiting a phase shift of π for a design energy of $E_0 = 27.3 \text{ keV}$. The absorption gratings G_0 and G_2 had Au structure heights of $h_0 = 42 \mu\text{m}$ and $h_2 = 26 \mu\text{m}$ with periods of $p_0 = 14 \mu\text{m}$ and $p_2 = 2 \mu\text{m}$, respectively. The distance between G_1 and G_2 was $d = 200 \text{ mm}$, corresponding to the 5th Talbot distance. The G_0 - G_1 distance was $l = 1400 \text{ mm}$, thus giving a total setup length of 1600 mm. The phase stepping was done by scanning the G_1 , with 9 stepping positions over 2 periods of the interference pattern for each scan. The noise in the images was increased by lowering the exposure time in each consecutive phase-stepping scan by a factor of two. Starting at 41 s exposure time per step, a total of 14 scans were performed, the exposure times thus varied from 41 s to 0.005 s per step, or from 369 s to 0.045 s for the total exposure time, respectively. Identical parameters were used to record the reference images. As samples, a square $10 \times 10 \text{ mm}^2$ PMMA (polymethyl methacrylate) bar and a square $5 \times 5 \text{ mm}^2$ Al bar were used. The PMMA bar was aligned along its diagonal, so that, in projection, it exhibits a phase shift like a symmetric triangle. The two bars were arranged in a way that leaves an empty area on one side of our field of view, allowing the analysis of image noise in the absence of a sample. The PMMA bar was used for the analysis of the noise properties in the differential phase-contrast channel and will not be discussed further. The experimental setup and sample arrangement can be seen in Fig. 1. The samples were positioned between G_0 and G_1 , 100 mm upstream of G_1 .

VI. RESULTS AND DISCUSSIONS

Analyzing the measured data for all exposure times by the discrete Fourier transform, as presented in Eqs. (2)–(4), the mean, amplitude, and visibility of the phase-stepping oscillation in each pixel were reconstructed. The reconstructed images in attenuation and dark-field contrast for four different exposure times can be seen in Fig. 3. Two rectangular regions-of-interest were chosen in the region of the strongly absorbing Al bar and the reference region to its right where no sample was positioned, as indicated in Fig. 3(a). The regions had a size of $34 \times 194 = 6596$ and $105 \times 194 = 20370$ pixels, respectively. The mean visibility in the reference was calculated from the measurement with the highest exposure time, resulting in a value of $\nu_r = 0.154 \pm 0.006$. The mean transmission of the Al bar was $T = 0.132 \pm 0.002$, while its mean dark-field signal was $D = 0.34 \pm 0.06$.

As we use a photon-counting x ray detector with an almost ideal, rectangular point-spread function, the response of each pixel can be interpreted as an independent statistical

FIG. 3. Experimental results for the sample arrangement shown in Fig. 1. In (a)-(d) transmission and (e)-(h) dark field contrast for four different exposure times. In (a), the Al and reference regions-of-interest are indicated by a solid and a dashed rectangle, respectively. As the SNR decreases, the dark-field signal rises above unity in areas in which attenuation is present, as visible in images (g) and (h).

process. The statistical behavior of the calculated parameters can thus be analyzed using a well-defined ensemble of pixels, as represented by the regions-of-interest. From the data measured in the regions-of-interest, the average and standard deviation over the area of each region were extracted and plotted against the exposure time. Since the exposure time, interpreted as a noise measure, depends on the utilized imaging system, we chose $\tilde{\sigma}$ as a reference parameter to quantify noise. This parameter can be calculated by the use of Eq. (11). For a source with constant flux F per pixel, the number of detected photons and, thus, the mean a_0 is proportional to the exposure time; thus the noise parameter $\tilde{\sigma}$ can be related to the exposure time per step t by $\tilde{\sigma} = 1/\sqrt{N} \cdot t \cdot F$. As the flux is unknown in the experiment, $\tilde{\sigma}$ was calculated by extracting the average number of detected photons from the region-of-interest in the reference scan.

Since the statistical behavior of the mean a_0 and amplitude a_1 implicitly enter into the calculation of the visibility, for shortness sake, we will limit the discussion of the measurement results to the visibility and the dark-field signal. The resulting plots for the mean visibility and its standard deviation as a function of $\tilde{\sigma}$ can be seen in Figs. 4(a) and 4(b), respectively. While the visibility is constant for low values of $\tilde{\sigma}$, it rises asymptotically toward the values predicted by the Rayleigh distribution in the limit of low SNR. This transition corresponds to a breakdown of information transfer from the raw images to the derived dark-field image.

The standard deviation σ_ν , as depicted in Fig. 4(b), changes its statistical behavior in a similar way. While the dependence of the visibility on the noise level can be described by Eq. (26) for high SNR, the dependence changes to Eq. (29) for low SNR.

The mean and median dark-field signal over the reference region-of-interest and the Al bar as a function of $\tilde{\sigma}$ are shown in Figs. 5(a) and 5(b), respectively. At low noise levels (region of high SNR), the dark-field signal is constant for both regions. For the Al bar, the rise of the signal with increasing noise level is not related to physical effects and, again, indicates a breakdown of information transfer. In the limit of low SNR, the median dark-field signal approaches $1/\sqrt{T} \approx 2.75$, as predicted by Eq. (35). In the absence of absorption the median remains constant with $E(D) = 1$, while the mean over a pixel ensemble increases without bound with increasing noise level. This deviation can be understood by the underlying statistics: in our calculations, we use averaging over a finite ensemble of pixels as an estimator for the expectation value $E(D)$. This estimator leads to good results as long as the statistical distributions are symmetric and fall off exponentially toward infinity, like the Gaussian PDF for high SNR. For low SNR, the distribution of the dark-field signal has a lower bound of zero, but is unbound and heavy-tailed toward high values. Thus, the estimation of the expectation value by averaging over a finite ensemble of pixels has a tendency to lead to values that are

FIG. 4. (a) Measured mean visibility for the reference region-of-interest (asterisks) and the Al bar (circles) and (b) its standard deviation $\sqrt{V}(\nu)$ as a function of the noise level $\tilde{\sigma}$. The solid and dashed lines describe the analytically expected values in the limit of high and low SNR, respectively. The rise of the visibility with increasing noise level indicates a breakdown of information transfer.

FIG. 5. Measured mean (asterisks) and median (circles) dark-field signal as a function of the noise level $\tilde{\sigma}$ for (a) the reference region and (b) the Al bar. The solid and dashed lines describe the analytically expected values in the limit of high and low SNR, respectively. In the limit of low SNR, the dark-field signal approaches a value related to the transmission of the sample, while it remains constant in the absence of a sample. The median is a better suited estimator for the expectation value than the mean, due to the skewness of the underlying distribution.

too high, due to the high variance of the underlying ratio distribution. Taking the median over the region-of-interest is a better estimator in this case, as the median is less sensitive to extreme outliers in the pixel ensemble. These results show that the physical meaning of the information presented in the dark-field image has to be interpreted with caution when high image noise is present. A fixed lower bound for the image noise, above which the reconstructed data are reliable, cannot be readily given, as the reconstruction errors depend on the transmission T and the dark-field signal D of the sample. Nevertheless, we can estimate an upper bound for the noise level $\tilde{\sigma}$ by demanding that the relative reconstruction errors of the dark-field signal in high SNR, as described by Eq. (32), have to be lower than the reconstruction errors for low SNR, described by Eq. (35). Equating both formulas yields the relation

$$\tilde{\sigma} \leq \nu_r \cdot \sqrt{\frac{4 - \pi}{\pi} \cdot \frac{TD^2}{TD^2 + 1}}. \quad (37)$$

For the dark-field signal of the Al-bar presented in Fig. 5, we thus obtain an upper bound of $\tilde{\sigma} \leq 0.01$. In the absence of a sample ($T = D = 1$), one obtains

$$\tilde{\sigma} \leq \nu_r \cdot \sqrt{\frac{4 - \pi}{2\pi}} \approx 0.37 \cdot \nu_r. \quad (38)$$

This corresponds to $\tilde{\sigma} \leq 0.057$ for the reference visibility presented in Fig. 4. The Eqs. (37) and (38) may therefore be used for an estimation of the minimum exposure time needed to obtain reliable dark-field data.

The results show that the approximation defined by Eqs. (24) and (25) can be used for the description of the noise properties of the dark-field signal without leading to major deviations from measurement results. The agreement of theory and measurement, as presented in Figs. 4 and 5, is convincing, so that the applicability of the analytical approach to describe the noise characteristics of image data in grating-based x ray dark-field images can be asserted.

VII. SUMMARY AND CONCLUSIONS

We have presented a short introduction into the signal and noise propagation in grating-based dark-field imaging. Starting from a Fourier transform-based image reconstruction approach, we analyzed the noise properties of the reconstructed

images and their limit for low SNR, identifying the Rician distribution as the PDF of the underlying statistics. The experimental results clearly show that the developed formalism describes the acquired data accurately. Theory and experiment show a breakdown of signal transmission in the limit of low SNR. In this limit, the dark-field signal is no longer connected to the coherence degradation, but rather to the attenuation in the sample. Thus, a lower limit for the SNR has to be imposed, below which the acquired data have to be handled and interpreted with caution. Nevertheless, above this limit, grating-based x ray dark-field imaging is able to deliver additional information about the examined sample and is well suited for application in medical imaging and nondestructive testing.

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