Quantum Optics Measurement Scheme for Quantum Geometry and Topological Invariants

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We show how a quantum optical measurement scheme based on heterodyne detection can be used to explore geometrical and topological properties of condensed matter systems. Considering a 2D material placed in a cavity with a coupling to the environment, we compute correlation functions of the photons exiting the cavity and relate them to the hybrid light-matter state within the cavity. Different polarizations of the intracavity field give access to all components of the quantum geometric tensor on contours in the Brillouin zone defined by the transition energy. Combining recent results based on the metric-curvature correspondence with the measured quantum metric allows us to characterize the topological phase of the material. Moreover, in systems where $S_z$ is a good quantum number, the procedure also allows us to extract the spin Chern number. As an interesting application, we consider a minimal model for twisted bilayer graphene at the magic angle, and discuss the feasibility of extracting the Euler number.

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Introduction.—Geometrical and topological properties of Bloch states play an important role in modern condensed matter physics [1,2]. A prominent manifestation of a global topological property is the quantized value of the Hall conductivity, as obtained by linear response calculations [3,4]. Quantum geometry, on the other hand, refers to quantities that are local in momentum space, such as the Berry curvature and quantum metric [5]. They are related to topology, but also influence the motion of electrons in the Brillouin zone (BZ) [6,7], nonlinear optical responses [8–13], and flat-band superconductivity [14,15].

Since the relation between geometry, topology, and observable quantities is often not obvious, it is important to identify experimental probes which allow us to measure these quantities, or at least provide relevant bounds [16]. To this end, several schemes utilizing linear response for the measurement of quantum geometry have been proposed [16–20]. However, while the spectroscopy of topological states of matter using semiclassical descriptions has been studied extensively [19–24], the quantum optics side is less explored. In quantum optics, entanglement between light and matter can lead to novel phenomena [25–28], and it can imprint properties of the matter system into the photon field. Since current-current correlation functions are fundamentally linked to the quantum metric [29,30], this suggests that the study of photon correlation functions of a cavity system provides a potentially fruitful avenue for probing a material’s geometrical and topological properties.

Here, we propose a quantum optical measurement scheme based on heterodyne detection [31,32]. The idea is to access general photon correlation functions inside a cavity, enabled by a coupling between the cavity and the environment, while a second photon field is superimposed on the field emitted from the cavity in order to slow down the time dependence of the signal. We demonstrate that such correlation functions can be directly related to the quantum geometric tensor, a quantity which encompasses both the Berry curvature and quantum metric. The latter is related to topology via the localization dichotomy [5,33,34]. Lastly, our method also provides an energy...
resolution which allows us to devise useful bounds on topological invariants of 2D systems.

Setup.—The heterodyne detection setup is sketched in Fig. 1(a). A Fabry-Perot cavity containing a 2D material is coupled to the environment, enabling an electric field to be transmitted through it while picking up signatures of the hybrid light-matter state within the cavity [35] [Fig. 1(b)]. (To avoid complications related to edge states, we may assume that the edges of the material are outside the cavity.) By placing a 50:50 beam splitter behind the output port and superimposing a second coherent laser source—henceforth referred to as the local oscillator (LO)—it is possible to slow down the signal emitted from the cavity. This allows us to bypass limitations in the time resolution of photodetectors, which is the goal of heterodyne detection [31,32,36], and enables the measurement of various photon correlation functions in the two detectors denoted by “D” in Fig. 1(a) [32,36,37].

We begin by describing the intracavity Hamiltonian which can be decomposed as \( \hat{H}_{\text{cav}} = \hat{H}_{\text{free}} + \hat{H}_J + \hat{H}_{\text{mat}} \). The free cavity field is described by \( \hat{H}_{\text{free}} = \hbar \Omega \hat{\alpha} \hat{\alpha} \), with \( \hat{\alpha} \) the photon annihilation operator and \( \Omega \) the cavity frequency, while \( \hat{H}_{\text{mat}} \) refers to the Hamiltonian of the electron system. The light-matter coupling in the Coulomb gauge and in the single-mode approximation reads

\[
\hat{H}_J = -\frac{q}{m} \hat{\alpha} \sum_{k,a} \hat{\epsilon}_{k,a}^\dagger \langle \psi_{k,a} \rangle \hat{p} \langle \psi_{k,a} \rangle \hat{\epsilon}_{k,a} + \frac{q^2}{2m} \hat{\alpha}^2 \sum_{k} \hat{\epsilon}_{k,a}^\dagger \hat{\epsilon}_{k,a},
\]

where \( q \) and \( m \) represent the electron’s charge and mass, respectively, \( \hat{\alpha} = \hbar \hat{\alpha}_f + \hat{\alpha}_f^* \) the vector potential treated within the dipole approximation, \( \hat{\alpha}_f \) the light-matter coupling parameter, and \( \hat{\alpha}_f = (\hat{\alpha}_f, \hat{\alpha}_f^*) \) the mode function of the vector potential. \( \hat{\epsilon}_{k,a} \) is the creation operator of an electron in the Bloch state \( |\psi_{k,a}\rangle \), and the material is represented in this basis by a noninteracting \( N \)-band model \( \hat{H}_{\text{mat}} = \sum_{k,a} \sum_{\omega=1}^N \epsilon_{\omega}(k) \hat{c}_{k,a}^\dagger \hat{c}_{k,a} \) with \( M \) occupied bands. \( |u_{k,a}\rangle = e^{-i k \hat{\alpha}} |\psi_{k,a}\rangle \) are solutions to the eigenvalue equation \( \hat{H}(k) |u_{k,a}\rangle = \epsilon_a(k) |u_{k,a}\rangle \). The light-matter coupling in the cavity gives rise to a hybrid matter-photon state, which manifests itself in the correlator of the intracavity photon mode,

\[
\langle \hat{\alpha}(t) \hat{\alpha}(t') \rangle - \langle \hat{\alpha}(t) \rangle \langle \hat{\alpha}(t') \rangle 
\approx \left( \frac{q \hbar}{\Omega} \right)^2 \sum_{\alpha=1}^M \sum_{\beta=1}^M e^{i \epsilon_{\alpha}(k) \alpha} f^\mu f^{\nu} A_\mu^{\alpha \beta}(k) A_\nu^{\beta \alpha}(k),
\]

where \( \langle \cdots \rangle \) is computed over the density matrix of the cavity according to the Gell-Mann low theorem [38] by adiabatically switching on \( \hat{H}_J \), and \( \epsilon_{\alpha}(k) \equiv (1/\hbar) [e_{\alpha}(k) - e_{\alpha}(k)] \) is the \( k \)-dependent valence-conduction-band gap.

\( t_{\text{rel}} \equiv t - t' \) and \( A^{\alpha \beta}_\mu(k) = \langle u_{k,\alpha} | i \partial_\mu | u_{k,\beta} \rangle \) (with \( \partial_\mu = \partial/\partial k^\mu \)) the non-Abelian Berry connection. We use the Einstein summation convention for the spatial indices \( \alpha, \beta \). Equation (2) neglects a higher-order term in \( \lambda \) and is valid in the regime \( \Omega \ll \min_{\omega} e_{\alpha}(k) \), i.e., away from topological transitions (see Supplemental Material [39], which includes Refs. [17,40–51], for details as well as realistic estimates of these parameters).

Our goal is to connect the intracavity correlator (2) to the detected photons in the setup of Fig. 1(a). To this end, we employ the theory of photodetection [32] and input-output theory [52], which yields

\[
\frac{n(t, \Delta t)n(t', \Delta t') - n(t, \Delta t) \cdot n(t', \Delta t')}{n(t, \Delta t)}
\approx \mathcal{D} \sum_{k} \sum_{\alpha=1}^M \sum_{\beta=1}^M \left[ e^{i \epsilon_{\alpha}(k) \alpha} f_{\alpha} \right] + \text{H.c.}
\times f^\mu f^{\nu} A_\mu^{\alpha \beta}(k) A_\nu^{\beta \alpha}(k)
\]

for the correlations between the photon counts \( n(t, \Delta t) \) at a single detector; see Supplemental Material [39]. The frequency of the local oscillator is \( \omega_L \) and coefficients related to the input-output theory and beam-splitter relations are subsumed into the coefficient \( \mathcal{D} \).

Equation (3) provides a direct link to the non-Abelian quantum geometric tensor (QGT),

\[
Q^{\alpha \beta}_\mu(k) = \sum_{\gamma=M+1}^N A^{\alpha \gamma}_\mu(k) A^{\gamma \beta}_\nu(k) = g^{\alpha \beta}_\mu(k) - \frac{i}{2} \mathcal{F}^{\alpha \beta \mu}_{\nu}(k),
\]

which can be decomposed into the quantum metric \( [g^{\alpha \beta}_\mu(k)] \) and Berry curvature \( [\mathcal{F}^{\alpha \beta \mu}_{\nu}(k)] \) contributions. In these expressions, \( \mu, \nu = x, y \) and \( \alpha, \beta \in [1, M] \). Since \( f^\mu f^{\nu} \sum_k \text{Tr}_b(Q^{\mu}_\alpha(k)) \) with \( \text{Tr}_b[\cdots] = \sum_{\alpha=1}^M [\cdots] \) can be measured by Eq. (3), we gain access to (i) the Chern number if \( f_{\alpha}(1) = \pm 1 \), (ii) general diagonal components of \( Q \) if \( f_{\alpha} = 1, 0 \) or \( 0, 1 \), and (iii) off-diagonal elements of \( g \) if \( f_{\pm} = \pm 1 \) in a way analogous to interband transitions driven by classical light [16,21,53,54].

For illustrative purposes, let us consider \( \omega_L = 0 \). Since \( e_{\alpha}(k) > 0 \), a Fourier transform of the first term on the right-hand side of Eq. (3) would select contours in the BZ with fixed energy difference \( \omega \) and sum up the values of \( f^\mu f^{\nu} A^{\alpha \beta}_\mu(k) A^{\gamma \beta}_\nu(k) \) at the corresponding \( k \)-points. This is shown in Fig. 1(c) for two different \( \omega \)’s represented by red and blue colors. The role of \( \omega_L \) is to lower the frequencies at which these resonances occur, consistent with the goal of heterodyne detection [31].

Band topology and localization dichotomy.—The positive semidefinite nature of \( Q^{\mu}_\alpha(k) \) implies certain inequalities involving the quantum metric and topological invariants [33,55]. The inability to devise a smooth gauge
of Bloch functions in a nontrivial topological phase is an obstruction to creating maximally localized Wannier functions [56]. Since the Wannier spread is directly related to the quantum metric [34], we will utilize this localization dichotomy [57] to relate Eq. (3) to topological invariants. From now on, we will use integrals whenever discussing topological invariants, but leave the discrete sums in Eq. (2) to enable a finite system description. For Chern insulators in 2D, the localization dichotomy manifests itself in bounds for the Chern number \( C = (1/2\pi) \int d^2k Tr_\beta [F_{xy}(k)] \) [33] as
\[
\pi |C| \leq \text{vol}_y \leq \text{vol}_g,
\]
where the so-called complexity of the band \( \text{vol}_g = \int d^2k \sqrt{\det \{ Tr_\beta [g(k)] \} } \) measures the BZ area with respect to the quantum metric [34] and \( \text{vol}_b = \sqrt{\int d^2k Tr_\beta [g_{xx}(k)] \int d^2k Tr_\beta [g_{yy}(k)] - \{ \int d^2k Tr_\beta [g_{xy}(k)] \}^2 } \).

Similarly, for the most commonly studied model of twisted bilayer graphene (TBG) with two occupied bands, it has been shown [55] that
\[
\frac{1}{4\pi} \int d^2k Tr_\beta [g_{xx}(k) + g_{yy}(k)] \geq |e_2|,
\]
where \( e_2 \) is the Euler number, a topological invariant found in models with \( C_2T \) symmetry [58,59]. Since \( \int d^2k Tr_\beta [g_{\mu\nu}(k)] \) can be determined by the photon correlation measurements, Eq. (3), we can in both cases provide an upper bound to the topological invariant.

**Inequalities involving the spin Chern numbers.**—While \( \mathbb{Z}_2 \) insulators have zero Chern number [60,61], it is interesting to ask whether our method can provide information on the spin Chern number, which we define as \( C_s = C_1 - C_\downarrow \). This is meaningful if the model Hamiltonian is of the form \( \hat{H}(k) = \text{diag}[h_\uparrow(k), h_\downarrow(k)] \equiv \text{diag}[h(k), \bar{h}(\bar{k})] \). For such a model with time reversal symmetry (TRS) and inversion symmetry (IS), we can prove the following chain of inequalities:
\[
\pi |C| \leq \pi (|C_\uparrow| + |C_\downarrow|) = 2\pi |C_\alpha| \leq \int d^2k \sqrt{\det \{ Tr_\beta [g(k)] \} } = 2 \int d^2k \sqrt{\det \{ Tr_\beta [g^{(\alpha\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\down arrow\downarrow\downarrow\down arrow\downarrow\downarrow\down arrow\downarrow\downarrow\down arrow\downarrow\downarrow\down arrow\downarrow\downarrow\down arrow\downarrow\downarrow\down arrow\downarrow\downarrow\down arrow\downarrow\downarrow\down arrow\downarrow\downarrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down arrow\down wheel
u = 2. The band touching at \((\pi, \pi)\) turns into a band touching at several momentum points once the spin degeneracy of the model is lifted, which subsequently contributes to a larger QGT. Nevertheless, we still find that \((1/2\pi)\text{vol}_g\) provides a useful upper bound of \(C_n\) away from the transition.

Three-band model.—Next, we analyze the three-band model from Ref. [33] with \(\tilde{H}_1(k) = -2t_{dd}(\cos(k_x) + \cos(k_y)) + \delta\), \(\tilde{H}_{12}(k) = \tilde{H}_{21}(k)\), \(\tilde{H}_{13}(k) = \tilde{H}_{31}(k)\), \(\tilde{H}_{23}(k) = 2t_{pp}\cos(k_y) - 2t'_{pp}\cos(k_x)\), \(\tilde{H}_{32}(k) = \tilde{H}_{31}(k)^*\), \(\tilde{H}_{33}(k) = 2t_{pp}\cos(k_y) - 2t'_{pp}\cos(k_x)\), \(t_{dd} = t_{pp} = t_{pp} = 1\), and \(\delta = -4t_{dd} + 2t_{pp} + \Delta - 2t_{pp}\Delta/(4t_{pp} + \Delta)\). With \(t_{pp}\Delta/(4t_{pp} + \Delta)\) and compute the QGT with respect to the lowest band. Figure 2(c) shows that similar to the QWZ model—\(I_{lb}\) closely follows \(\text{vol}_g\) over the entire trivial phase, and to some extent also in the nontrivial phase. Since \(\sqrt{\det(g)}\) can differ from \(|F_{xy}|/2\) in models with more than two bands, \(I_{lb}\) is not a good estimate of \(\text{vol}_g\). Still, \(I_{lb}\) can be used to estimate the location of the topological transition with high accuracy.

Twisted bilayer graphene.—As a further application, we consider the four-band tight-binding model of TBG from Ref. [55],

\[
\tilde{H}(k) = \mu_\xi(\Delta \sigma_0 + \xi \sigma_z) + \mu_0 \rho(k) \cdot \sigma - 2\lambda \mu_\xi \sigma_z f(k),
\]

where \(\mu(0, x, y, z)\) and \(\sigma(0, x, y, z)\) are (identity matrices) Pauli matrices acting in the space of orbital and sublattice degrees of freedom, respectively. \(\rho_{1,2}(k) = \sum_{i=1}^{3}\{r \cos(\sin(\delta_i \cdot k) + t' \cos(\sin(-2\delta_i \cdot k))\}, \rho_3(k) = 0\), and \(f(k) = \sum_{i=1}^{3}\sin(d_i \cdot k)\). With \(a_{1,2}\) representing the real-space moiré lattice unit vectors, we have the nearest-neighbor vectors \(\delta_1 = 1/3 a_1 + \frac{2}{3} a_2\), \(\delta_2 = -\frac{1}{3} a_1 - \frac{2}{3} a_2\), \(\delta_3 = \frac{1}{2} a_1 + \frac{\sqrt{5}}{2} a_2\), and the second-nearest-neighbor vectors \(d_1 = a_1, d_2 = a_2, d_3 = -a_1 - a_2\). In order to make the bands as flat as possible, Xie et al. chose \(t' = -t/3\), \(\lambda = (2/\sqrt{27})t\), and \(\Delta = 0.15t\) with \(t = 1\) [55]. A nonzero \(\xi\) opens a gap between the otherwise degenerate occupied and unoccupied bands at \(K\), rendering the topological phase trivial [see Fig. 3(a)].

FIG. 3. Geometrical quantities for TBG plotted as a function of \(t'/t\). The trivial phase is indicated with a blue shading; whenever one of the plotted quantities drops below the light-blue line, Eq. (6) implies \(e_0 = 0\). (a) Band structure in the trivial phase where the nodal point at \(K\) is gapped due to \(\xi = 0.8\) (encircled points). (b) Trivial phase \((e_0 = 0)\) with \(\xi = 0.8\). The inset shows the Wilson loop in the moiré Brillouin zone for \(t'/t = -0.4\) (solid) and \(t'/t = 1\) (dashed) which are gapped for \(\xi = 0.8\). (c) Nontrivial phase \(|e_0| = 1\) with \(\xi = 0\) and unit winding of the Wilson loop for all values of \(t'/t\).
The nontrivial topology in this model manifests itself in Wilson loop winding, which is seen as a crossing in the loop diagrams shown in the insets of Fig. 3 [58,63]. In the same figure, we present geometrical quantities for the two phases of the model. In view of Eq. (6), Fig. 3(b) shows that for most values of $f/f'$, we can determine whether the system is in a topologically trivial state by looking at momentum integrals over $g_{ii}$ ($i = x, y$).

In the nontrivial case depicted in Fig. 3(c), we see the same trend of $g_{ii}$ increasing with increasing bandwidth, but notice a remarkable agreement between $(1/4\pi) \int d^2k Tr [g_{xx} + g_{yy}]$ and $|e_2|$ at smaller bandwidths, which is consistent with Ref. [18]. For the models considered in Ref. [33], the saturation of such inequalities was found to be related to a small bandwidth or band-gap ratio, which is consistent with the results in Fig. 3. We also observe that $(1/2\pi)vol_j$ almost perfectly for all bandwidths in the nontrivial phase. Although the relationship between $(1/2\pi)vol_j$ and $|e_2|$ is unclear, we numerically demonstrate that $(1/2\pi)vol_j$ provides an upper bound to $|e_2|$ for all $f/f'$.

**Conclusions.**—We devised a way of extracting quantum geometry, and indirectly topology, by means of a cavity QED setup combined with a heterodyne detection scheme and the localization dichotomy. The power of this scheme was demonstrated with applications to paradigmatic models hosting interesting geometry and topology. Utilizing the energy resolution of the method we provided improved markers of the system’s geometry. A future prospect is the application of our scheme to 3D materials (thin films) with nontrivial geometrical properties. While the Chern number is not directly determined by the Berry curvature in 3D, the delocalization of Wannier orbitals can be detected. Furthermore, in systems with Berry curvature dipoles and zero Chern number, the metric-curvature correspondence still enables a characterization of the Berry curvature dipole strength [64,65]. While the present study assumed weak light-matter coupling, strong light-matter interactions provide a pathway for engineering novel topological phases [66]. In a future project, it would thus be interesting to adapt Eq. (3) to the study of geometrical and topological properties of such hybrid light-matter states.

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