Puzzles in the hadronic contributions to the muon anomalous magnetic moment

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ABSTRACT: We summarize recent developments in the Standard-Model evaluation of the anomalous magnetic moment of the muon a_μ, both in the hadronic-light-by-light and hadronic-vacuum-polarization contributions. The current situation for the latter is puzzling as we are confronted with multiple discrepancies that are not yet understood. We present updated fits of a dispersive representation of the pion vector form factor to the new CMD-3 data set and quantify the tensions with the other high-statistics $e^+e^- \to \pi^+\pi^-$ experiments in the contribution to a_μ in the energy range up to 1 GeV, as well as in the corresponding contribution to the intermediate Euclidean window. See ref. [1] for an extended version of this contribution.

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Contents

1 Introduction

The anomalous magnetic moment of the muon $a_\mu = (g - 2)_\mu / 2$ has received a lot of attention over the past years, as the current experimental value [2, 3]

$$a_\mu^{\text{exp}} = 116\,592\,061(41) \times 10^{-11} \tag{1.1}$$

already has an impressive precision of 0.35 ppm and further improvements from the Fermilab experiment are expected in the near future. In order to fully exploit this progress on the experimental side, the theoretical prediction within the Standard Model (SM) needs to achieve a similar level of precision. The experimental value (1.1) is in 4.2σ tension with the SM prediction

$$a_\mu^{\text{WP}} = 116\,591\,810(43) \times 10^{-11} \,, \tag{1.2}$$

as published in the 2020 White Paper (WP) [4]. The theoretical uncertainty is completely dominated by hadronic effects, in particular by hadronic vacuum polarization (HVP), which in eq. (1.2) is determined via dispersion relations and experimental input on the photon-inclusive $e^+ e^- \to$ hadrons cross sections. The WP result (1.2) has been challenged by the first lattice-QCD result achieving sub-percent precision [5]. Moreover, the new $e^+ e^-$ data set on the two-pion channel by CMD-3 [6] differs significantly from the input used in the WP.

Here, we summarize the current status of the SM prediction for a_μ, which now needs to address a multitude of discrepancies and tensions (see ref. [7]). We focus on the hadronic contributions, describing recent progress on the sub-leading hadronic light-by-light (HLbL) scattering in section 2, before discussing several aspects of HVP in section 3.

2 Hadronic light-by-light scattering

The HLbL contribution to a_μ is determined by the hadronic four-point function of electromagnetic currents

$$\Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3) = -i \int d^4x \, d^4y \, d^4z \, e^{-i(q_1 \cdot x + q_2 \cdot y + q_3 \cdot z)} \langle 0|T\{j_{\text{em}}^\mu(x) j_{\text{em}}^\nu(y) j_{\text{em}}^\lambda(z) j_{\text{em}}^\sigma(0)\}|0\rangle \,, \tag{2.1}$$

where $q_4 = q_1 + q_2 + q_3$. A tensor decomposition

$$\Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3) = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma} \Pi_i(s, t, u), \tag{2.2}$$

into a redundant set of 54 Lorentz structures was derived in refs. [8, 9], in such a way that the structures $T_i^{\mu\nu\lambda\sigma}$ individually satisfy the Ward-Takahashi identities, while at the same time the scalar functions Π_i are free of kinematic singularities and zeros, leading to a master formula for the HLbL contribution to a_μ directly in terms of the hadronic scalar functions Π_i,

$$a_\mu^{\text{HLbL}} = \frac{2\alpha^3}{3\pi^2} \int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \sqrt{1 - \tau^2} Q_1^3 Q_2^3 \sum_{i=1}^{12} T_i(Q_1, Q_2, \tau) \bar{\Pi}_i(Q_1, Q_2, \tau), \tag{2.3}$$

where T_i are known integration kernels and only 12 independent linear combinations $\bar{\Pi}_i$ of the Π_i contribute. Further, the decomposition (2.2) allowed us to set up a dispersive framework for HLbL in four-point kinematics, which enabled the evaluation of the dominant contributions to HLbL with controlled and much reduced uncertainties, in particular of the pseudoscalar-pole and two-pion contributions. The uncertainty on HLbL is now dominated on the one hand by the contributions of hadronic resonances in the $(1-2)$ GeV range, which were estimated using hadronic models, on the other hand by the matching to short-distance constraints (SDCs) that follow from the operator-product expansion (OPE). Therefore, since the publication of the WP, efforts were directed towards an improved evaluation of these sub-dominant contributions, addressing scalar resonances beyond the $f_0(500)$ [10], axial-vector resonances [11–14], tensor resonances and D-waves [15, 16], which require a modification of the dispersive framework itself [17], as well as the SDCs on HLbL [12, 18–20]. Work is in progress to combine all these developments, together with improvements of the η, η' pole contributions [21–24], into a full data-driven evaluation of a_μ^{HLbL}.

Since the WP publication, the HLbL contribution has also been evaluated within lattice QCD with competitive uncertainties. The results of refs. [25–27] are compatible with the phenomenological WP value, but point to a slightly larger central value. In order to meet the final experimental precision goal, the uncertainties in HLbL should be further reduced to the level of about 10%, which seems feasible for both phenomenological and lattice-QCD evaluations. However, all these improvement in the HLbL evaluation will only have a real impact once the tensions in the evaluations of HVP are resolved, to which we turn next.

3 Hadronic vacuum polarization

The discrepancy between the experimental value for a_μ (1.1) and the SM evaluation (1.2) is reduced to only 1.5σ if the evaluation of HVP is replaced by [5]

$$a_\mu^{\text{HVP, LO, BMWc}} = 7\,075(55) \times 10^{-11}. \tag{3.1}$$

However, this number is in 2.1σ tension with the WP evaluation based on e^+e^- cross-section data. The resolution of this tension is crucial in order to update the WP prediction and to reach a single competitive SM prediction for a_μ [28]. The current puzzling situation has triggered intense scrutiny of both the lattice and dispersive evaluations. So-called window quantities, obtained by introducing

Discrepancy	$a_\mu^{\pi\pi}\big|_{[0.60,0.88]\,\mathrm{GeV}}$	$a_\mu^{\pi\pi}\big|_{\leq 1\,\mathrm{GeV}}$	int window
SND06	1.8σ	1.7σ	1.7σ
CMD-2	2.3σ	2.0σ	2.1σ
BaBar	3.3σ	2.9σ	3.1σ
KLOE″	5.6σ	4.8σ	5.4σ
BESIII	3.0σ	2.8σ	3.1σ
SND20	2.2σ	2.1σ	2.2σ
Combination	$4.2\sigma\,[6.1\sigma]$	$3.7\sigma\,[5.0\sigma]$	$3.8\sigma\,[5.7\sigma]$

Table 1. Significance of the discrepancies between fits to CMD-3 and the other experiments, taking into account the correlations due to the systematics in the dispersive representation, as well as the χ^2 inflation of the fit errors. For the combined fit, the discrepancies in square brackets exclude the systematic effect due to the BaBar-KLOE tension.

weight functions in the Euclidean-time integral of the coordinate-space representation of HVP [29], have proved useful, as the intermediate window is much less affected by lattice systematics than the entire HVP contribution to a_μ. The BMWc value for this quantity is in 3.7σ tension with the cross-section data [30] and several lattice collaborations have now confirmed this result [31–34].

The information from the window quantities as well as the constraints on the hadronic running of α imply that the differences mainly come from the region below $\approx 2\,\mathrm{GeV}$ [35–40]. In the low-energy region, the two-pion channel completely dominates and modifications of the cross-section data can be confronted with the constraints of analyticity and unitarity on the pion vector form factor (VFF) [39]. We are using the representation for the VFF [41]

$$F_\pi^V(s) = \Omega_1^1(s) \times G_\omega(s) \times G_{\mathrm{in}}^N(s)\,, \tag{3.2}$$

where $\Omega_1^1(s)$ denotes the Omnès function with the elastic $\pi\pi$-scattering P-wave phase shift $\delta_1^1(s)$ as input, $G_\omega(s)$ accounts for the resonantly enhanced isospin-breaking ρ–ω interference effect, and further inelastic contributions are parametrized by a conformal polynomial $G_{\mathrm{in}}^N(s)$. Although this dispersive representation only depends on a few parameters, there is enough freedom to describe all major experiments individually — in particular, the constraints of analyticity and unitarity do not resolve the tension between BaBar [42] and KLOE [43], see ref. [41]. Similarly, even modifications of the cross-section data well beyond the BaBar-KLOE tension can be accommodated by the dispersive constraints, with rather uniform shifts in the two-pion cross section leading to a correlated shift in the pion charge radius [39], which potentially could provide an independent cross check if an improved lattice determination of the pion charge radius became available. Exactly such a shift in the cross-section data is indeed realized in the recent measurements by CMD-3 [6]. In ref. [1], we present updated results for the fit of the dispersive representation (3.2) to the major experiments, including CMD-3. We find a p-value of 20% for the fit to CMD-3: the data are compatible with the dispersive constraints. Our representation also allows us to quantify the tension to the other experiments for the full energy range up to 1 GeV, shown in table 1. As noted in ref. [44], SND20 [45] is the only experiment that cannot be fit with a good p-value, while BaBar, KLOE, BESIII [46], SND06 [47], and CMD-2 [48] do permit acceptable fits.
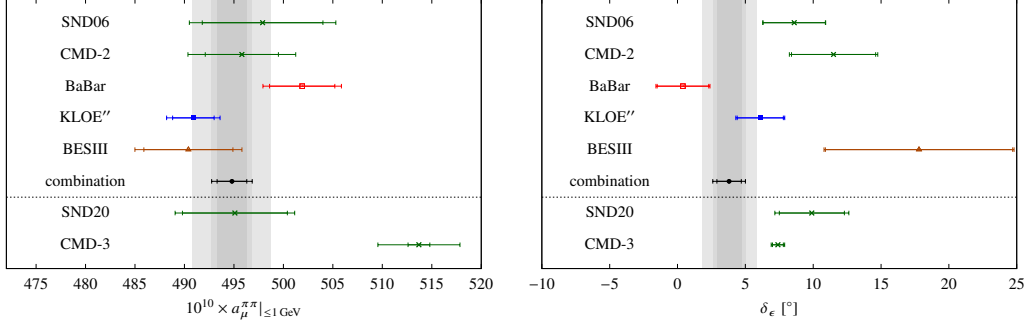
Figure 1. Left: results for $a_\mu^{\pi\pi}$ in the energy range $\leq 1\,\mathrm{GeV}$. Right: results for the phase of the ρ–ω mixing parameter, δ_ϵ. The smaller error bars refer to the fit uncertainties, inflated by $\sqrt{\chi^2/\mathrm{dof}}$, the larger error bars to the total uncertainties. The gray bands correspond to the combined fit to NA7 and all e^+e^- data sets apart from SND20 and CMD-3, with the largest band including the additional systematic effect due to the BaBar-KLOE tension.

The discrepancies among the different e^+e^- experiments are shown in figure 1. For a_μ itself, the discrepancy between CMD-3 and the combination of the other experiments by far exceeds the BaBar-KLOE tension or the one between BMWc and the WP, amounting to 5σ for the HVP integral up to $1\,\mathrm{GeV}$ and even more around the ρ peak or in the intermediate window. Further tensions are visible directly in the fit parameters, e.g., the complex phase δ_ϵ of the ρ–ω mixing parameter ϵ_ω, an observable generated by radiative channels such as $\rho \to \pi^0\gamma \to \omega$ [44], differs widely among the experiments.

4 Conclusions

The evaluation of the hadronic contributions to a_μ has been the subject of intense research efforts. While recent work on HLbL promises to reach the precision goal set by the Fermilab experiment, the interpretation of the SM prediction is currently complicated by the presence of a multitude of puzzles in the HVP contribution: the disagreement between lattice QCD and hadronic cross sections on the one hand, but also a new discrepancy between CMD-3 and all other $e^+e^- \to \pi^+\pi^-$ experiments on the other. We presented updated fit results of our dispersive representation of the pion vector form factor to the $e^+e^- \to \pi^+\pi^-$ data sets. The fit to the CMD-3 data did not reveal any conflict with the dispersive constraints, yet the discrepancies to the other experiments are substantial: when compared to the combination they amount to 5σ for the entire energy range below $1\,\mathrm{GeV}$, even more for some partial quantities, see table 1. Forthcoming results from ongoing $e^+e^- \to \pi^+\pi^-$ analyses, a reinvestigation of radiative corrections, and further lattice-QCD computations scrutinizing the BMWc result for the full HVP contribution will be indispensable to understand the current puzzling situation.

Acknowledgments

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