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By C. JACCARD

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Four-Point Method for Measuring the Volume and Surface Conductivities of a Thin Sample

By C. JACCARD, Federal Institute for Snow and Avalanche Research,
7260 Weissfluhjoch/Davos, Switzerland

1. Introduction

A four-point method for measuring the anisotropy of resistivity has been described by P. SCHNABEL (Philips Res. Repts., vol. 19, pp. 43-52, 1964). On a thin sample with two flat parallel surfaces, two electrodes A and B are brought on one side and two other ones C and D just opposite on the other side. An electric current flows first between A and C and the potential difference between B and D is measured; then, the current flows between A and B and the potential is measured between C and D .

If the material is isotropic, the resistances $R_1 = V_{BD}/I_{AC}$ and $R_2 = V_{CD}/I_{AB}$ allow to determine any two of the following physical parameters: volume conductivity, electrode spacing or sample thickness; in the case of electric anisotropy, two components of the conductivity tensor can be obtained.

We attempted to apply this method to measure the DC conductivity tensor of ice, but the obtained results were incompatible with known properties of the material. The discrepancy has been attributed with success to a large surface conductivity. Schnabel's method can be extended to obtain from the two measured resistances the bulk and the surface conductivities simultaneously. The purpose of this paper is to describe such a method. A correction term is introduced to account for the surface current and is then included into the general formula relating the measured resistances with the conductivities. The sensitivity of the method and the influence of a finite sample surface are evaluated.

The surface conductivity of ice crystals has been determined according to this method (C. JACCARD: Electrical conductivity of the Surface Layer of Ice, Proceedings of the International Conference on Snow and Ice, Sapporo 1966, to be published). The results have been compared with direct measurements on thin samples, extrapolated to zero thickness. The agreement is well within the experimental error and this proves the usefulness of the method described here.

2. Potential with Surface Current

Let us consider a system of cylindrical coordinates ϱ, ψ, z with the upper half space $z > 0$ filled with an homogeneous and isotropic material having a bulk electric conductivity σ_v and a surface conductivity σ_s in the plane $z = 0$. At the origin ($z = 0, \varrho = 0$) a current J is injected into the material. As there are no other current sources and as the bulk conductivity is a constant, the electric potential φ must satisfy in the

upper half space to the Laplace equation:

$$\Delta\varphi = 0 \quad (z > 0) \quad (1)$$

and must vanish at infinity:

$$\varphi = \infty, \quad z = \infty : \varphi = 0. \quad (2)$$

On the surface, the bulk current $\mathbf{j}$ has a normal component j_z produced by the divergence of the surface current $\mathbf{j}_s$:

$$\text{div}_s \mathbf{j}_s + j_z = 0 \quad (z = 0). \quad (3)$$

Introducing the characteristic length

$$\lambda = \frac{\sigma_s}{\sigma_v} \quad (4)$$

this condition can be expressed for the potential by

$$\frac{\lambda}{\varrho} \frac{\partial}{\partial \varrho} \left(\varrho \frac{\partial \varphi}{\partial \varrho} \right) + \frac{\partial \varphi}{\partial z} = 0 \quad (z = 0) \quad (5)$$

as the symmetry is obviously cylindrical. The solution of the system (1, 2, 5) takes then the form:

$$\varphi = \left(\frac{J}{2\pi\sigma_v} \right) [(z^2 + \varrho^2)^{-1/2} - \lambda z (z^2 + \varrho^2)^{-3/2}]. \quad (6)$$

The first term corresponds to the applied current source and the second one to a current dipole with a strength $J\lambda$ with the source outside ($z = -0$) and the sink inside ($z = +0$) the material. The dipole field accounts for the finite surface conductivity and vanishes with it, as it is proportional to the length λ .

3. Potentials and Currents in a Flat Sample

Let us consider now an infinite sample with parallel flat surfaces and a thickness w situated in the original coordinate system between $z = 0$ and $z = -w$. Four point electrodes are brought on the surface at the points A ($z = 0, \psi = 0, \varrho = 0$), B ($z = 0, \psi = 0, \varrho = s$), C ($z = -w, \psi = 0, \varrho = 0$), and D ($z = -w, \psi = 0, \varrho = s$). In a first configuration, the current is injected into the sample at A and flows out at C , and the potential is measured between B and D . To account for the boundary conditions we superpose an infinity of virtual current sources on the z -axis with the potential given by Equation (6), the positive ones at the ordinates $z = \pm 2nw$ and the negative ones at $z = \pm (2n+1)w$ with $n = 0, 1, 2, 3 \dots \infty$. The sources and sinks must have a strength of $\pm 2J$: at point A for example, half of the current flows into the real plate but the other half flowing into the superposed virtual plate is not taken into account in the current measurement. The current dipoles have to be replicated in an analogous way. Formula (6) applied to this system yields then the following value for the potential difference:

$$V_B - V_D = \left(\frac{J_{AC}}{\pi w \sigma_v} \right) \left[G(\alpha) + \left(\frac{\lambda}{w} \right) G_1(\alpha) \right] \quad (7)$$

with

$$\alpha = \frac{s}{w}. \quad (8)$$

The function $G(\alpha)$ has already been given by SCHNABEL and has the value

$$G(\alpha) = \alpha^{-1} - 2 \sum_{n=1}^{\infty} (-1)^{n+1} (\alpha^2 + n^2)^{-1/2} \quad (9)$$

and the function $G_1(\alpha)$ is

$$G_1(\alpha) = 2 \sum_{n=1}^{\infty} (-1)^{n+1} n (\alpha^2 + n^2)^{-3/2}. \quad (10)$$

In the second configuration, the current is injected at A and flows out of the sample at B , and the potential is measured between C and D . The virtual current sources are then taken at $z = \pm 2nw$, with $\varrho = 0$ for the positive ones, $\varrho = s$ for the negative ones and $\psi = 0$ for both, with a corresponding distribution of the dipoles. The potential difference is then

$$V_C - V_D = \left(\frac{J_{AB}}{\pi w \sigma_v} \right) \left[H(\alpha) - \left(\frac{\lambda}{w} \right) H_1(\alpha) \right], \quad (11)$$

where $H(\alpha)$ is also one of Schnabel's functions:

$$H(\alpha) = 2 \sum_{n=1}^{\infty} \{ (2n-1)^{-1} - [\alpha^2 + (2n-1)^2]^{-1/2} \} \quad (12)$$

and

$$H_1(\alpha) = 2 \sum_{n=1}^{\infty} \{ (2n-1)^{-2} - (2n-1) [\alpha^2 + (2n-1)^2]^{-3/2} \}. \quad (13)$$

The resistances as defined in Section 1 are then given by

$$R_1 \pi w \sigma_v = G(\alpha) + \left(\frac{\lambda}{w} \right) G_1(\alpha), \quad (14)$$

$$R_2 \pi w \sigma_v = H(\alpha) - \left(\frac{\lambda}{w} \right) H_1(\alpha). \quad (15)$$

Solving these expressions yields the following formulae for the conductivities:

$$\pi w \sigma_v R_1 = \frac{F_1}{[F_2 + (R_2/R_1)]}, \quad (16)$$

$$\pi \sigma_s R_1 = \frac{F_3 [F_4 - (R_2/R_1)]}{[F_2 + (R_2/R_1)]^2}, \quad (17)$$

$$\frac{\lambda}{w} = \left(\frac{F_3}{F_1} \right) \frac{[F_4 - (R_2/R_1)]}{[F_2 + (R_2/R_1)]} \quad (18)$$

with

$$F_1(\alpha) = \frac{(G H_1 + G_1 H)}{G_1}, \quad (19)$$

$$F_2(\alpha) = \frac{H_1}{G_1}, \quad (20)$$

$$F_3(\alpha) = \frac{(G H_1 + G_1 H) G}{G_1^2}, \quad (21)$$

$$F_4(\alpha) = \frac{H}{G}. \quad (22)$$

Numerical values of G and H have been tabulated by SCHNABEL for $\alpha = 0,05$ to $2,0$ in $0,05$ steps. The two other functions can be obtained easily from the function $Q(\alpha)$

$$Q(\alpha) = \sum_{n=1}^{\infty} n (\alpha^2 + n^2)^{-3/2}. \quad (23)$$

For $\alpha \leq 2$, the term for $n \geq 6$ can be developed in α^2 with a fast convergence. The formula

$$\left. \begin{aligned} Q(\alpha) &= \sum_{n=1}^5 n (\alpha^2 + n^2)^{-3/2} + \sum_{m=0}^{\infty} (-1)^m \left(\frac{\alpha}{2}\right)^{2m} (2m+1)! (m!)^{-2} \sum_{n=6}^{\infty} n^{-2(m+1)} \\ &= \sum_{n=1}^5 n (\alpha^2 + n^2)^{-3/2} + 0.18132 - 0.01183 \left(\frac{\alpha}{2}\right)^2 \\ &\quad + 0.00115 \left(\frac{\alpha}{2}\right)^4 - 0.00012 \left(\frac{\alpha}{2}\right)^6 + 0.00001 \left(\frac{\alpha}{2}\right)^8 \end{aligned} \right\} \quad (24)$$

gives Q with an accurate 4th place after the decimal point. The functions G_1 and H_1 are related with Q by

$$G_1(\alpha) = 2 Q(\alpha) - Q\left(\frac{\alpha}{2}\right), \quad (25)$$

$$H_1(\alpha) = \frac{3}{2} Q(0) - 2 Q(\alpha) + \frac{1}{2} Q\left(\frac{\alpha}{2}\right). \quad (26)$$

From the function (for $\alpha \leq 2$)

$$\left. \begin{aligned} M(\alpha) &= 2 \sum_{n=1}^{\infty} [n^{-1} - (n^2 + \alpha^2)^{-1/2}] \\ &= 2 \sum_{n=1}^5 [n^{-1} - (n^2 + \alpha^2)^{-1/2}] + 0.06558 \left(\frac{\alpha}{2}\right)^2 \\ &\quad - 0.00319 \left(\frac{\alpha}{2}\right)^4 + 0.00023 \left(\frac{\alpha}{2}\right)^6 - 0.00002 \left(\frac{\alpha}{2}\right)^8 \end{aligned} \right\} \quad (27)$$

the functions G and H can be calculated according to the expressions

$$G(\alpha) = \frac{1}{\alpha} - 2 \ln 2 + M(\alpha) - M\left(\frac{\alpha}{2}\right), \quad (28)$$

$$H(\alpha) = M(\alpha) - \frac{1}{2} M\left(\frac{\alpha}{2}\right). \quad (29)$$

4. Sensitivity of the Method

The sensitivity depends upon the relative importance of the surface and of the volume contributions to the overall sample conductance. If the surface conductivity plays a minor role; i.e. if $\lambda/w \lesssim 0,1$ (for $\alpha \simeq 1$), the bulk conductivity can be obtained by any of the expressions (14) and (15). The relative error $\Delta\sigma_v/\sigma_v$ is then the smallest of the sums $\Delta w/w + \Delta R_1/R_1$ or $\Delta w/w + \Delta R_2/R_2$. In this case, the ratio R_2/R_1 is near the value of F_4 and σ_s or λ have a correspondingly small absolute value according to Equation (17) or (18). The absolute error is small, but the relative one is large, and

only the order of magnitude can be determined with certainty. The most favorable situation occurs when both contributions to the conductance are of the same order. The sensitivity is then roughly the same for σ_v and for σ_s or λ , about $\Delta w/w + \Delta R_{1,2}/R_{1,2}$. If the surface conductivity plays the major role, e.g. if $\lambda/w \gtrsim 1$ (for $\alpha \simeq 1$), the ratio R_2/R_1 is near the value $-F_2$ and σ_s , being the inverse of a small number, cannot be determined accurately, and neither does σ_v .

Another electrode configuration gives better results for σ_s . A very thin sample bears on one of its sides four collinear electrodes: A and D (separation $s \ll w$) for the current, and B and C (separation $s' < s$) for the potential, with equal distances AB and CD . The resistance $R' = V_{BC}/I_{AD}$ is then given by the expression

$$2\pi\sigma_s R' = \ln \left[\frac{(s + s')}{(s - s')} \right] \quad (30)$$

and the relative error on σ_s is of the order of $\Delta s/s + \Delta s'/s' + \Delta R'/R'$.

5. Error Produced by a Finite Sample Radius

The calculations of Section 3 apply strictly only to a sample with an infinite extension perpendicular to the z -axis. Let us examine what happens when the sample is circular with a finite radius R . The boundary condition is

$$\frac{\partial \varphi}{\partial \varrho} = 0 \quad \text{for} \quad \varrho = R. \quad (31)$$

As the potentials produced by the dipoles resulting from the surface conductivity have a smaller range than those of the main current sources and sinks, they are affected at a lesser degree by condition (24). Therefore, as we need only the order of magnitude of the error, we may disregard the surface conductivity.

We take a source and a sink on the upper surface of the disc shaped sample ($z = 0$, $\varrho = s/2$, $\psi = 0$ and π respectively). The unperturbed potential (up to a multiplicative constant)

$$\varphi_1 = \sum_{n=-\infty}^{+\infty} \left\{ \left[\varrho^2 + \frac{s^2}{4} + (z - 2nw)^2 + \varrho s \cos \psi \right]^{-1/2} - \left[\varrho^2 + \frac{s^2}{4} + (z - 2nw)^2 - \varrho s \cos \psi \right]^{-1/2} \right\} \quad (32)$$

has to be corrected by an harmonic term φ_2 , periodic in ψ , with the general form

$$\varphi_2 = \sum_{\nu=0}^{\infty} \sum_{\mu=1}^{\infty} A_{\nu\mu} \cos(\nu\psi) \cos\left(\frac{\mu\pi z}{w}\right) I_{\nu}\left(\frac{\mu\pi\varrho}{w}\right) \quad (33)$$

where the I_{ν} are modified Bessel functions. The coefficients $A_{\nu\mu}$ are determined by condition (24) which, with

$$B_{\nu\mu} = A_{\nu\mu} \left(\frac{\mu\pi}{w}\right) I'_{\nu}\left(\frac{\mu\pi R}{w}\right), \quad (34)$$

takes the form:

$$\sum_{\nu=0}^{\infty} \sum_{\mu=1}^{\infty} B_{\nu\mu} \cos(\nu\psi) \cos\left(\frac{\mu\pi Z}{w}\right) = - \left[\frac{\partial}{\partial \varrho} \varphi_1(z, \varrho, \psi) \right]_{\varrho=R} = F(z, R, \psi). \quad (35)$$

The $B_{\nu\mu}$ can be calculated exactly by a Fourier inversion, but to estimate their order of magnitude, only an upper limit for F is necessary. This limit is obtained in replacing $\cos\psi$ by -1 , z by $+w$, in the first term of φ_1 , and $\cos\psi$ by $+1$, z by $-w$ in the second term. The sums, which are in fact integrals over stair like functions, are replaced by integrals over the smooth functions enveloping the stairs, from above or from below according to the sign. With the hypothesis $R - s/2 > w$, the integral limits can be replaced by ± 1 and the end result is

$$|F| < F_M = \frac{17}{Rw}. \quad (36)$$

The larger of the $B_{\nu\mu}$ is obviously B_{11} which is estimated to be

$$B_{11} = \left(\frac{1}{\pi w}\right) \int_{-\pi}^{+\pi} \int_{-w}^{+w} F \cdot \cos\psi \cos\left(\frac{\pi z}{w}\right) d\psi dz < 4 F_M = \frac{68}{Rw} \quad (37)$$

and therefore

$$A_{11} < \left(\frac{68}{\pi R}\right) / I'_1\left(\frac{\pi R}{w}\right). \quad (38)$$

In the region of interest, $|\varphi_1|$ can be approximated by $1/w$ and the relative correction is then

$$\left|\frac{\varphi_2}{\varphi_1}\right| \lesssim \frac{68 (w/R) I_1(\pi s/2 w)}{I'_1(\pi R/w)}. \quad (39)$$

If this estimation is done for the configuration in which the source and the sink are on opposite sides of the sample, the same order of magnitude is found.

In most of cases, the separation s is about the same as the sample thickness w , so that $I_1(\pi s/2 w) \simeq I_1(\pi/2) \simeq 1$. But if the sample radius is at least equal to four times the thickness, the argument of I'_1 in the denominator is at least 12, and the

Table I. Values of the functions $Q, G, H, G_1, H_1, F_1, F_2, F_3, F_4$ of the parameter x

x	Q	G	H	G_1	H_1	F_1	F_2	F_3	F_4
0,1	1,629	8,623	0,01044	1,617	0,03007	0,1708	0,01860	0,9109	0,001211
0,2	1,583	3,649	0,04091	1,537	0,1160	0,3163	0,07549	0,7501	0,01121
0,3	1,513	2,023	0,08898	1,416	0,2464	0,4409	0,1740	0,6297	0,04399
0,4	1,426	1,242	0,1512	1,270	0,4061	0,5483	0,3198	0,5360	0,1218
0,5	1,332	0,8015	0,2239	1,113	0,5796	0,6414	0,5209	0,4620	0,2793
0,6	1,235	0,5325	0,3032	0,9568	0,7542	0,7229	0,7882	0,4023	0,5695
0,7	1,141	0,3601	0,3861	0,8111	0,9207	0,7949	1,135	0,3529	1,072
0,8	1,053	0,2465	0,4701	0,6804	1,074	0,8591	1,578	0,3112	1,907
0,9	0,9730	0,1701	0,5533	0,5665	1,211	0,9170	2,138	0,2754	3,252
1,0	0,9005	0,1182	0,6345	0,4695	1,332	0,9697	2,837	0,2440	5,369
1,1	0,8356	0,08248	0,7128	0,3883	1,438	1,018	3,703	0,2163	8,642
1,2	0,7778	0,05781	0,7880	0,3209	1,529	1,163	4,766	0,1916	13,63
1,3	0,7264	0,04064	0,8597	0,2654	1,608	1,106	6,060	0,1693	21,15
1,4	0,6806	0,02866	0,9279	0,2200	1,677	1,146	7,622	0,1493	32,38
1,5	0,6397	0,02025	0,9927	0,1829	1,736	1,185	9,493	0,1312	49,02
1,6	0,6030	0,01434	1,054	0,1527	1,788	1,222	11,71	0,1148	73,51
1,7	0,5701	0,01018	1,113	0,1280	1,833	1,259	14,33	0,1001	109,4
1,8	0,5404	0,007231	1,169	0,1078	1,873	1,294	17,38	0,08682	161,6
1,9	0,5135	0,005146	1,222	0,09128	1,908	1,329	20,90	0,07493	237,4
2,0	0,4891	0,003666	1,272	0,07769	1,939	1,364	24,96	0,06435	347,0

function itself at least 20000, because it grows rapidly, like an exponential. The correction is then at most 0,02% and is quite negligible. If a correction smaller than 1% is required, the sample radius has to be larger than three times the thickness, which is easy to obtain practically.

This favorable situation is due to the presence of I'_1 in the denominator in the formula for A_{11} , and this results from the boundary condition (24). If we assume that the sample is enclosed in a highly conducting ring, we have to require that the potential itself vanishes on the boundary, not its differential with respect to the radius. This does not let appear the function I'_v and the corrective term has to be considered for much larger values of the ratio R/w . Thus the sample circumference must not be grounded.

Résumé

Un échantillon mince d'épaisseur w , avec une conductivité électrique de volume et de surface finie, a sur un de ses côtés deux électrodes ponctuelles A et B (avec une séparation s) et vis à vis sur l'autre côté, deux autres électrodes C et D . Un courant passe d'abord entre A et C , ensuite entre A et B ; une différence de potentiel est mesurée entre B et D , ensuite entre C et D . Avec les résistances V_{BD}/I_{AC} et V_{CD}/I_{AD} , on peut calculer la valeur de la conductivité de surface et de volume à l'aide de fonctions universelles du paramètre s/w .

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