A mechanically-based model of snow slab and weak layer fracture in the Propagation Saw Test

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Abstract

Dry-snow slab avalanche release is the result of failure initiation in a weak snowpack layer buried below a cohesive snow slab, which is then followed by rapid crack propagation. The Propagation Saw Test (PST) is a field experiment which allows to evaluate the critical crack length for the onset of crack propagation and the propagation distance. Although a widely used method, the results from this field test are difficult to interpret in practice because (i) the fracture process in multilayer systems is very complex and only partially explored and (ii) field data is typically insufficient to establish direct causal links between test results and snowpack characteristics. Furthermore, although several studies have focused on the critical crack length assuming linear elasticity for the slab, it still remains unclear how the complex interplay between the weak layer failure and slab fracture impacts the outcome of the PST.

To address this knowledge gap, an analytical model of the PST was developed, based on the Euler-Bernoulli beam theory, in order to compute both the critical crack length and the propagation distance as a function of snowpack properties and beam geometry (e.g. beam length and slab height). This work aims to create a link between the two main outcomes of the PST, namely full propagation (END) and slab fracture (SF), and the quantitative results of critical crack length and propagation distance.

Moreover, introducing empirical relationships based on laboratory experiments (Scapozza, 2004; Sigrist, 2006) between the elastic modulus, the tensile strength and slab density, it is possible to describe the onset of slab fracture for a given geometry of the PST using only the slab density. As a result, the model allows to reproduce the...
increasing trend of the propagation distance with increasing slab density, as observed in field experiments. For slabs characterized by low density, slab fracture occurs before reaching the critical crack length (SFb); for intermediate density values, slab fracture occurs after the onset of crack propagation in the weak layer (SFa); then, large densities lead to full propagation in the weak layer without slab fracture (END).

**Keywords:** Propagation Saw Test, Mechanical Model, Snow Mechanics, Fracture
1. Introduction

Dry-snow slab avalanche release is generally caused by failure initiation in a weak layer buried below cohesive snow slabs, which is then followed by rapid crack propagation (Schweizer et al., 2003). The Propagation Saw Test (PST) is an experimental in-situ technique that has been introduced to assess crack propagation propensity. The PST, developed simultaneously by Sigrist and Schweizer (2007) and Gauthier and Jamieson (2008), involves isolating a conventional volume of snow in the downslope direction (Figure 1). Once the weak layer has been identified in the stratigraphy (e.g., from a manual snow profile or a compression test), a saw is used to cut through it progressively. If the crack length from the sawing reaches a critical value, crack propagation can occur. Depending on the snowpack properties, the crack within the weak layer can induce slab fracture (SF) or propagate further to the end of the column, which is called full propagation (END).

Statistical studies over large sets of field data have confirmed the relevance of the PST, highlighting a relatively high correlation between the test results and the likelihood of avalanche release, despite the number of false alarms (Simenhois and Birkeland, 2009; Ross and Jamieson, 2012; Reuter et al., 2015). Recent PST measurements have shown that the bending of the slab is a key element for the onset of crack propagation (van Herwijnen et al., 2010; van Herwijnen et al., 2016) and the subsequent dynamic regime. Numerous experiments were performed by van Herwijnen and Jamieson (2005) and van Herwijnen and Birkeland (2014), utilizing high speed cameras and particle tracking velocimetry (PTV), which have revealed important details of the intricate relationship between the propagation of the crack in the weak layer and the slab deformation field.

The seminal work of McClung (1979) provided a first modeling of the shear failure observed in the weak layer and it has been improved, at a later time, by McClung (2003), Chiaia et al. (2008) and Gaume et al. (2013). However, observations of remote triggering of avalanches as well as the bending of the snow slab observed in field PSTs (van Herwijnen et al., 2010; van Herwijnen et al., 2016) have challenged these past theories. Consequently, this led to the development of models focusing on the weak
layer collapse, such as the anticrack model of Heierli et al. (2008) or the dynamic shear collapse model of Gaume et al. (2015, 2017) which accounts for the mixed-mode failure of the weak layer (Reiweger et al., 2015).

Frequently, inconsistencies in results can be given by the unclear interrelations among snowpack properties (Gaume et al., 2017) and the difficulty to treat complex crack propagation phenomena in multilayered systems (Hutchinson and Suo, 1991; Habermann et al., 2008). In addition, most of the existing studies have focused solely on the critical crack length for the onset of crack propagation in the weak layer (assuming linear elasticity for the slab as in LEFM), but thus disregarding its influence on the fracture propensity of the slab. Hence, on the one hand, this approach does not allow a thorough understanding and a quantitative evaluation of the complex interplay between crack propagation in the weak layer and slab fracture. On the other hand, there is a lack of scientific evidence on how this interdependence can affect the outcome of the PST.

In order to address this knowledge gap, this work aims to provide a quantitative analysis of the PST, accounting for the most important elements, specifically the interaction between the weak layer and slab fracture, as well as the frictional contact occurring during slab bending. The proposed model is based on the Euler-Bernoulli beam theory, which allows to compute the stress state of the snowpack during the PST, given the crack length, geometrical and mechanical properties. With the introduction of this approach, it is possible to compute the critical crack length and the propagation distance and define the outcome of the test in terms of full propagation (END) and slab fracture (SF) outcomes.

The paper is organized as follows. First, a qualitative mechanical analysis of the three possible PST results (full propagation, slab fracture and fracture arrest) was conducted. Based on these cases, the mechanical model considering the interaction of the slab and the weak layer was studied. With the Euler-Bernoulli beam theory and the brittle failure assumption, the stress state of the two layers of interest were analyzed separately. The model was applied to a realistic case study in which snowpack properties were interrelated and, then, model predictions were compared with field data. A detailed parametric study was also performed. Finally, the results are discussed with respect to previous studies highlighting advantages and limitations of the approach in
2. Phenomenology of the PST fracture modes

The set-up of the test includes an isolated volume of snow of $b = 30$ cm for the width and at least 1 meter in length $l_{\text{tot}}$ in the downslope direction (Figure 1). The full depth of the snowpack is considered in the experiment. The case study is composed of a snow slab overlying a rigid substratum (of total length $l_{\text{tot}}$) with a weak layer in between. Figure 2 shows the side view of the idealized model, where the horizontal is parallel to the slope plane with angle $\psi$.

Once the saw reaches the critical crack length $l_c$, the crack can propagate in the weak layer, which is followed by a possible slab fracture. The PST results are commonly reported with the following cases (Gauthier and Jamieson, 2008):

- **Full propagation (END)**: the crack propagates through the whole weak layer.
Figure 2: Side view sketch of the modeled Propagation Saw Test.

- **Slab fracture (SF):** the crack propagation in the weak layer is stopped by a sudden fracture in the slab.

- **Fracture arrest without slab fracture (ARR):** the propagation in the weak layer stops to a point that is difficult to identify.

The results of the test consist of the critical crack length and the propagation distance.

The first question to address is the sequence of events that could take place in a Propagation Saw Test. The slab represents a gravitational load on the weak layer. At the beginning of the test, increasing the crack length $l$ creates a volume of snow that is supported on one side while hanging freely on the other one and the weak layer has a reduced resisting area (Figure 3(a)). The layer is displaced both vertically and horizontally under its own weight and might touch the rigid substratum (Figure 3(b)). The crack length $l$ at which this initial contact (IC) is observed is identified with $l = L_{IC}$. Initially, only the lower corner of the beam free end (henceforth called tip) rests on the rigid substratum. Therefore, the resulting effect is a hinged restrain, where the beam does not displace vertically, but has freedom to rotate (Figure 3(c)). Then, following the increase of $l$, the slab bends back due to its own weight and rests with vertical cross section with respect to the rigid substratum. At this point, not only is the vertical movement constrained but the rotation of the beam is also fixed. The crack length $l$ required for this condition to occur is called the length of full contact (FC).
and it is denoted by $L_{FC}$ (Figure 3(d)). If the sawing continues, then the contact zone of the slab and the substratum layer increases. However, the length between the saw and the first touching point remains constant, being equal to the full contact length $L_{FC}$ (Figure 3(c)) due to equilibrium requirements. The beam is now considered as a double supported beam, with a fixed length of $L_{FC}$, and, consequently, with a linear bending moment and constant shear on the cross sections. Notably, for typical snowpack condition, cases (c), (d) and (e) are rarely observed if the crack length (created by the saw) is lower than the critical one. However, the frictional contact is important during the dynamic phase of the crack propagation in the weak layer (Gaume et al., 2015) and, consequently, for slab avalanche release.

Then, the possible failure modes that can appear during the test are discussed. We consider the resulting stresses on the slab $\sigma_s$ and on the weak layer $\sigma_w$, with respect to their limit values $\sigma_s^y$ and $\sigma_w^y$ in a generic sense (i.e. either shear, tension or compression). With different values of the ratios $R_s = \sigma_s / \sigma_s^y$ and $R_w = \sigma_w / \sigma_w^y$, it is possible to define various limiting cases. Their values are limited between zero, representing a stress free situation, and one, corresponding to failure. The variation of $R_s$ and $R_w$ influences the way the slab interacts with the weak layer, as well as the the crack propagation process that follows.

2.1. $R_s \approx 1$ and $R_w << 1$: low strength of slab

The first limiting case presents a slab which has not yet developed enough strength. On the contrary, the weak layer is relatively strong with respect to the load. In this case, slab fracture occurs in relation to a very short crack length without creating a cantilever structure and crack propagation in the weak layer is very unlikely. In this case, the critical crack length tends to the total length ($l_c \rightarrow l_{tot}$), since the saw has to cut the whole specimen in order to release the entire slab.

2.2. $R_s << 1$ and $R_w \approx 1$: low strength of weak layer

In this case, the weak layer has a low strength, whereas the slab strength is high and any initial crack created by the saw provokes failure of the weak layer and crack propagation. Since the stresses in the weak layer can only increase during the test,
Figure 3: Stages of the Propagation Saw Test: after initial bending (a), first contact of slab and rigid sub-stratum is reached at crack length $l = L_{IC}$ (b). Following the sawing (c), the full contact length is achieved for $l = L_{FC}$ (d), at which point the cross section has rotated back to vertical. Successively, the length of the beam under bending is kept constant at $L_{FC}$ for $l > L_{FC}$, while the contact zone is increasing as the crack further progresses (e).
Figure 4: Example of slab fracture outcomes in the PST: (a) crack propagation in the weak layer followed by a slab fracture (SFa); (b) slab fracture due to high bending and tensile stresses before crack propagation in the weak layer (SFb).

the propagation takes place along the whole specimen as soon as the crack is initiated. Consequently, the critical crack length tends to zero ($l_c \to 0$).

2.3. $R_s \approx 1$ and $R_w \approx 1$: slab and weak layer close to failure

In this case, the slab and weak layer stresses are comparable with their respective strengths leading to a competitive mechanism of failure between the slab and weak layer.

If the slab stress ratio is lower than the weak layer ratio ($R_s \leq R_w$), then the crack propagation can be observed in the weak layer upon reaching the critical crack length. Within this context, the propagation may stop when stresses in the slab reach its strength causing a fracture and subsequently releasing the gravitational load (Figure 4(a)). Conversely, if the weak layer stress ratio is lower than the slab ratio ($R_s \geq R_w$), the slab may fracture prior to any propagation in the weak layer (Figure 4(b)).

This set of different phenomena shows the importance of analyzing and predicting the conditions at which each single mechanism occurs. A detailed analysis of the material properties and resulting stress states can help to reveal the failure mechanism which causes the observed result in the PST.
3. Mechanical model

The objective of this section is to describe analytically the stresses and failure conditions for the slab and the weak layer under quasi-static conditions.

The isolated volume of snow for the test is presented in Figure 5 in 3D. The total length of the specimen is $l_{\text{tot}}$, while $l$ represents the crack length and, consequently, $l_{w} = l_{\text{tot}} - l$ is the length of the remaining weak layer.

The slab is assumed to consist of a single homogeneous material, with constant width $b$ and height $h$. A coordinate system $x, y, z$ is placed in the center of mass of the orthogonal cross section at the crack tip. There, the stresses distribution arising in the slab is studied in the case of a free end boundary condition similar to a field PST. Furthermore, the end of the slab at $x = l$ is subjected to various boundary conditions depending on its interaction with the other layers.
The weak layer has a thickness $h_w$ and width $b$. It is assumed that the weak layer reacts uniformly to the external load at the interface with the slab, where a coordinate system is designated on the geometrical center of this interface, pointing in the upward direction $\eta$. Finally, a perfectly rigid substratum is located under the weak layer.

For the purposes of the mechanical analysis, the material composing each layer of the snowpack is assumed to be homogeneous and isotropic. The interface between the layers is considered capable of perfectly transferring the load and maintaining the cross section plane upon deformation. This assumption is required in order to study independently the slab and the weak layer with the beam theory. Subsequently, the failure modes observed in the test are introduced. The critical crack length for the onset of crack propagation in the weak layer is defined as $l_w^c$ whereas the critical crack length for slab fracture is $l_s^c$. In addition, the signed crack propagation distance $l_p = l_w^c - l_s^c$, for which slab fracture occurs after weak layer failure, is also computed. It is important to note that negative values of $l_p$ represent the case of slab fracture appearing before any propagation in the weak layer.

### 3.1. Mechanical model of the slab

The slab is studied as a homogeneous beam of length $l$, with width $b$ and height $h$. The load due to gravity is uniformly distributed along the entire beam. Each unitary-length cross section provides a constant load $q$ with vertical and horizontal components according to the slope angle $\psi$, as depicted in Figure 6.

The uniformly distributed load due to the weight of the slab is:

$$ q = \rho ghb $$

where $\rho$ is the density of the slab and $g$ is the gravitational acceleration. The horizontal and vertical components of the load due to gravity are:

$$ q_h = q \sin \psi \quad q_v = q \cos \psi $$

The vertical load, orthogonal to the centerline, causes bending, which rotates the cross sections and lowers the tip of the slab (Figure 6(b)). The horizontal load, parallel to
the beam centerline, causes stretching along the coordinate $x$ in the downslope direction (Figure 6(c)). In general, these two effects are not independent, but as long as the displacements remain small, the hypothesis of superposition of effects holds. Hence, the beam is studied separately for each load case and, then, the results are superimposed. To describe the variation of the internal forces with respect to the load and to the geometric characteristics, the Euler-Bernoulli beam theory is used.

A beam that is subjected to the set of forces previously presented usually experiences failure due to bending moment, shear, normal tension or a combination of these (Figure 7). It is important to note that for slender beams, shear stress is significantly smaller than the stress given by bending moment and tensile force. For this reason, we disregard any shear effect in the slab.

3.1.1. Bending stresses

In the Euler-Bernoulli theory, the behavior of an elastic beam under a distributed load $q_v$ is described by the following ordinary differential equation:

$$\frac{d^2}{dx^2} \left( EI \frac{d^2 v(x)}{dx^2} \right) + q_v = 0$$

(3)
in terms of the vertical displacement function $v$ and the horizontal coordinate $x$. Once $v(x)$ is computed, then it is possible to derive the rotation $\theta$, the curvature $\chi$, the bending moment $M$ and the shear force $T$ along the beam:

$$
\theta = \frac{dv(x)}{dx} \quad \chi = \frac{d\theta(x)}{dx} \quad M = -EI\chi \quad T = \frac{dM(x)}{dx} \frac{dT(x)}{dx} + q_v(x) = 0 \quad (4)
$$

13 where $E$ is the elastic modulus and $I$ is the second order moment of inertia of the cross section. To solve this ordinary differential equation, four boundary conditions have to be set. Two of them are imposed on the cross section at the crack tip ($x = 0$), where vertical displacement and rotation are assumed to be null. Likewise, the other two conditions are obtained by considering the free end of the beam ($x = l$). At the beginning of the PST, the beam is free to rotate (bending moment equal to zero) and displace vertically (shear force equal to zero) at its tip. This situation is denoted Bending Scheme I and depicted in Figure 8(a).

The solution for the vertical displacement function of the cantilever is:

$$
v_I(x) = \frac{-q_v(6l^2x^2 - 4lx^3 + x^4)}{24E_sI_s} \quad (5)
$$

24 The highest moment is found at the crack tip, corresponding with the clamped end $x = 0$:

$$
M_I = \frac{q_v l^2}{2} \quad (6)
$$

26 When the tip displaces vertically to a height equal to $h_w$, it touches the rigid sub-stratum. $L_{IC}$ is the beam length at which this occurs and it is computed as:

$$
L_{IC} = \sqrt{\frac{8E_sI_s h_w}{q_v}} \quad (7)
$$

13
For \( l \geq L_{IC} \) the slab is in contact with the rigid substratum and the previous bending scheme is not valid anymore. In the Bending Scheme II, the cross section at the crack tip \( (x = 0) \) is assumed to be fixed, whereas the vertical motion is restrained at the other end \( x = l \) (Figure 8(b)). The small part that is in contact with the substratum is reacting with a vertical force \( F_t \), but it does not apply any bending moment to restrain the rotation of the cross section at the tip.

The resulting vertical displacement function \( v(x) \) is expressed as:

\[
v_{II}(x) = -\frac{3h_wx^2}{2l^2} - \frac{q_i^2x^2}{16E_iI_i} + \frac{h_wx^3}{2l^3} + \frac{5q_ilx^3}{48E_iI_i} - \frac{q_ix^4}{24E_iI_i}
\]  

(8)
The maximum value of the bending moment is located at the clamped end:

\[ M_{II}(0) = \frac{qvl^2}{8} \left( 1 + \frac{3L_{IC}^4}{l^4} \right) \]  

The force \( F_t \) represents the reaction of the rigid substratum at the contact point with the slab and it reduces the stresses in the slab as well as in the weak layer:

\[ F_t = T(l) = \frac{3ql}{8} - \frac{3E_sI_shw}{l^3} = \frac{3ql}{8} \left( 1 - \frac{L_{IC}^4}{l^4} \right) \]  

With further increase of length \( l \), the free end of the beam is rotating back to zero due to its weight. Consequently, the Bending Scheme II is valid until the end section of the beam is orthogonal to the rigid substratum. This length is defined as the full contact length \( L_{FC} \). Subsequently, all slab in excess of \( L_{FC} \) will be supported by the rigid substratum. To find the full contact length \( L_{FC} \), \( \theta(l) = 0 \) and therefore:

\[ L_{FC} = \sqrt{\frac{72E_sI_shw}{q_c}} = \sqrt{3}L_{IC} \]  

which shows that the crack length difference between the beginning of Bending Scheme II and the beginning of Bending Scheme III is \( L_{FC} - L_{IC} = (\sqrt{3} - 1)L_{IC} \approx 0.75L_{IC} \).

Here, it is important to recall the concepts presented in Section 2. By increasing the crack length beyond the full contact length (\( l > L_{FC} \)), both ends of the beam are restrained in rotation, i.e., the beam must be horizontal at the boundaries. At the same time, in this scheme, the contact zone between the slab and the rigid substratum increases with increasing crack length as the length of the beam subjected to bending is always \( L_{FC} \) (for a crack length \( l > L_{FC} \)). In fact, the tip of the beam at \( x = L_{FC} \) is in contact (\( v(L_{FC}) = -h_w \)) and parallel (\( \theta(L_{FC}) = 0 \)) to the rigid substratum and the exceeding crack length is supported directly by the rigid substratum. Consequently, this scheme is also characterized by a frictional force exerted by the weight of the slab on the substratum. This configuration is denoted as the Bending Scheme III.

In the bending scheme III, the boundary conditions on a beam of constant length \( L_{FC} \) are:

\[ v_{III}|_{x=0} = 0 \quad v_{III}|_{x=L_{FC}} = -h_w \]  

\[ v'_{III}|_{x=0} = 0 \quad v'_{III}|_{x=L_{FC}} = 0 \]
This yields the vertical displacement function:

\[ v_{III}(x) = \frac{h_w}{L_{FC}^3} \left(2x^3 - 3L_{FC}x^2\right) - \frac{q_v x^4}{24E_s I_s} (x - L_{FC})^2 \]  

(14)

Once again, the maximum moment is located at the crack tip \( x = 0 \), and reads:

\[ M_{III}(0) = \frac{q_v L_{FC}^2}{6} \]  

(15)

It is important to highlight that the value of the bending moment at the cross section of the crack tip will not increase for longer crack lengths \( l \), since, as it was previously discussed, the length of the active beam remains constant at the full contact length \( L_{FC} \), while the rest of the slab is directly supported by the underlying substratum.

### 3.1.2. Normal stresses

Beside the bending moment, which allows to derive the characteristic lengths \( L_{IC} \) and \( L_{FC} \), the slab is subjected to a horizontal force \( N \) exerted in slope direction.

To describe it, two traction schemes are introduced (Figure 9):

- **Simply supported:** This case is linked with the Bending Schemes I and II, where the tip of the beam is either free or only touches the rigid substratum at a single point, not exerting enough contact force to engage friction. It is assumed that the cross section at the crack tip \( x = 0 \) is fixed with respect to horizontal displacements. Given \( q_h = q \sin(\psi) \), the horizontal force is:

\[ N = q(l - x) \sin(\psi) \quad \text{for } l < L_{FC} \]  

(16)

- **Simply supported with additional frictional resistance:** This case occurs when the beam is in full contact with the rigid substratum, i.e. \( l > L_{FC} \), the friction between the slab and the rigid substratum is helping to sustain the horizontal load. The friction force depends on the weight of the part in contact \( w_{air} \) and on the static friction coefficient \( k_f \) between the slab and the rigid substratum. Since \( w_{air} \) is assumed to be zero up to the full contact length \( L_{FC} \), the second horizontal force scheme represents the fixed-fixed roller mechanism, corresponding to Bending Scheme III. Then, it is possible to compute the horizontal force as:

\[ N = q(l - x) \sin(\psi) - k_f qbh (l - L_{FC}) \cos(\psi) \quad \text{for } l \geq L_{FC} \]  

(17)
3.1.3. Slab failure criterion

In order to determine the crack length that causes fracture in the slab, the stresses in the resisting cross section are computed. The resisting cross section is defined as the area of the beam, orthogonal to the center line, above the crack tip. As it was discussed before, the highest value of bending moment and horizontal force was observed in this zone. In the elastic range, to evaluate the distribution of stresses (normal to the cross section) due to bending moment \( M \) and normal traction \( N \), it is possible to use Navier’s formula:

\[
\sigma(y) = \frac{M}{I_s} y + \frac{N}{A_s}
\]

(18)

where the second moment of inertia \( I_s \) and the area of the section \( A_s \) are

\[
I_s = \frac{bh^3}{12}, \quad A_s = bh
\]

(19)

Navier’s formula gives the distribution of stresses along the height of the beam given the bending moment \( M \) and the horizontal force \( N \).

In this model, the fracture of the slab is assumed to be brittle and to appear due to tensile stresses. The value \( \sigma_{st} \) represents the tensile strength (i.e., the maximum admissible tensile stress) for the snow in this layer.

Hence, failure is reached when the top surface of the slab reaches the stress \( \sigma_{st} \) due to the bending moment \( M \) and the horizontal force \( N \):

\[
\sigma_{st} = \frac{M}{I_s} \frac{h}{2} + \frac{N}{A_s}
\]

(26)

As the tensile strength is reached, i.e. \( \sigma_t^e = \sigma_{st} \), detachment of the slab is predicted.

Table 1 outlines the applied bending moment \( M \) and horizontal force \( N \) for each force scheme previously presented, as well as the final expression of \( \sigma_t^e \), the tensile...
Table 1: Summary of the forces and the stress functions computed at the resisting section of the slab.

<table>
<thead>
<tr>
<th>Scheme I: $l \leq L_{IC}$</th>
<th>External Forces: $M = \frac{q_{l} l^2}{2}$, $N = q_{l} l$ (20)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Tensile stress: $\sigma_{t} = \rho g h \left[ \frac{l}{h} \sin (\psi) + \frac{3 l^2}{h^2} \cos (\psi) \right]$ (21)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Scheme II: $L_{IC} &lt; l \leq L_{FC}$</th>
<th>External Forces: $M = \frac{q_{l} l^2}{8} \left( 1 + \frac{3 L_{IC}^2}{l^2} \right)$, $N = q_{l} l$ (22)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Tensile stress: $\sigma_{t} = \rho g h \left[ \frac{l}{h} \sin (\psi) + \frac{3 l^2}{4 h^2} \cos (\psi) \left( 1 + \frac{3 L_{IC}^2}{l^2} \right) \right]$ (23)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Scheme III: $l &gt; L_{FC}$</th>
<th>External Forces: $M = \frac{q_{l} L_{IC}^2}{2} = \frac{q_{l} L_{FC}^2}{6}$, $N = q_{l} l - k_{f} q_{l} (l - L_{FC})$ (24)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Tensile stress: $\sigma_{t} = \rho g h \left[ \frac{l}{h} \sin (\psi) - k_{f} \frac{l - L_{FC}}{h} \cos (\psi) + \frac{L_{FC}^2}{h^2} \cos (\psi) \right]$ (25)</td>
</tr>
</tbody>
</table>
stress on the top surface of the slab, as a function of the crack length. By enforcing 
\( \sigma_s^t = \sigma_{s,0}^t \) (i.e. tensile failure), it is possible to invert these expressions in order to 
determine the critical crack length for slab fracture \( l_c \). Moreover, the variation of the 
stresses as function of the crack length can be studied.

3.2. Mechanical model of the weak layer

In this study, a homogeneous weak layer of finite thickness is considered. Recent 
studies have highlighted the mixed-mode shear-compression failure behavior of weak 
snowpack layers (Reiweger et al., 2015; Chandel et al., 2015). A simplified failure 
criterion with constant compressive and shear strength, \( \sigma_w^c \) and \( \tau_w^s \), respectively, is 
introduced as the only mechanical characterization of the weak layer (Figure 10).

![Figure 10: Weak layer failure modes: (a) crushing due to compression and (b) shear failure due to the combined effect of horizontal and vertical load.](image)

The stresses in the weak layer are described for the load schemes presented in 
Figure 10, which are representative of the three force schemes in the slab. In order 
to evaluate the stresses in the weak layer as a function of the crack length, the weight 
of the slab is considered to be concentrated in its center of gravity. This assumption 
holds if there is a perfect transfer of forces between the two layers and if the slab 
is approximated as a rigid solid when computing the forces on the weak layer. The 
distance \( l_M \) between the center of gravity of the slab and the center of the resisting 
section in the weak layer (as previously identified in Figure 5) is given by geometric
considerations as:
\[ l_M = \frac{l}{2} \quad \text{for} \quad l < L_{IC} \]
\[ l_M = \frac{l}{2} \quad \text{for} \quad L_{IC} \leq l < L_{FC} \]  
\[ l_M = \frac{L_{FC}}{2} \quad \text{for} \quad l \geq L_{FC} \]  
In addition to the weight of the slab, it is also necessary to recall \( F_t \), the vertical reaction force applied at the tip of the beam which is caused by the contact of the slab with the rigid substratum. When \( l < L_{IC} \), the vertical force \( F_t \) is zero since there is no contact. Instead, when \( L_{IC} \leq l < L_{FC} \), the force \( F_t \) increases with increasing crack length. Finally, in the case of \( l \geq L_{FC} \), \( F_t \) is constant because the active beam is always \( L_{FC} \) long.

Given the uniformly distributed gravitational load \( q \), the vertical \( W_v = q l \) and horizontal \( W_h = q_h l \) components of the slab weight result in shear, compression and bending moment on the weak layer, which are summarized in Table 2. Schemes II and III include the reduced vertical weight of the slab due to the presence of the force \( F_t \).

The stresses in the weak layer can be calculated with Navier’s and approximate shear expressions:
\[ \sigma_w(\eta) = \frac{N_w}{A_w} + \frac{M_w}{I_w} \zeta \quad \tau_w = \frac{T_w}{A_w} \]  
where \( \zeta \) is the coordinate along the longitudinal dimension of the weak layer (Figure 5), \( I_w \) and \( A_w \) are respectively the second order of inertia and the area of the resisting section:
\[ I_w = \frac{b (l_{tot} - l)^3}{12} \quad A_w = b (l_{tot} - l) \]

The stress in the weak layer is limited by the maximum compression at the crack tip or by the maximum shear force at the interface between the layers.

Table 2 summarizes the computed shear and compressive stresses applied to the weak layer. A dependence is observed on the inverse of the quantity \( l_{tot} - l \), which results in a nonlinear increase of stresses, growing to infinity for \( l \to l_{tot} \).

Enforcing that the compressive stress \( \sigma_w \) and shear stress \( \tau_w \) in the weak layer have to be equal to their respective strength values \( \sigma_{yc} \) and \( \tau_{ys} \), it is possible to compute the
Figure 11: Equilibrium schemes for the computation of stresses in the weak layer: (a) case for $l < L_{IC}$; (b) case for $L_{IC} \leq l \leq L_{FC}$; (c) case for $l > L_{FC}$. CG is the center of gravity of the slab not supported by the part in contact with the rigid substratum, $W_h$ and $W_v$ are respectively the horizontal and vertical components of the slab weight, $l_M$ is the distance of CG from the center of the reacting weak layer and $F_t$ is the vertical reaction given by the contact between slab and rigid substratum.
Table 2: Summary of the forces acting on the weak layer.

<table>
<thead>
<tr>
<th>Scheme I: $l \leq L_{IC}$</th>
<th>External forces:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_w = W_n$</td>
<td>$T_w = W_h$</td>
</tr>
<tr>
<td>$M_w = W_n \frac{l}{2} + W_h \frac{h}{2}$</td>
<td>(28)</td>
</tr>
<tr>
<td>Compressive $\sigma_c$ and shear $\tau_s$ stress:</td>
<td></td>
</tr>
<tr>
<td>$\sigma_c = \rho gh \cos(\psi) \frac{h}{(\tan(\psi))} \left[1 + \frac{3(l+h \tan(\psi))}{(l+2h)}\right]$</td>
<td>(29)</td>
</tr>
<tr>
<td>$\tau_s = \rho gh \sin(\psi) \frac{h}{(\tan(\psi))}$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Scheme II: $L_{IC} &lt; l \leq L_{FC}$</th>
<th>External forces:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_w = W_n - F_t$</td>
<td>$T_w = W_h$</td>
</tr>
<tr>
<td>$M_w = W_n \frac{l}{2} + W_h \frac{h}{2} - F_t \frac{l}{2}$</td>
<td>(30)</td>
</tr>
<tr>
<td>Compressive $\sigma_c$ and shear $\tau_s$ stress:</td>
<td></td>
</tr>
<tr>
<td>$\sigma_c = \rho gh \cos(\psi) \frac{h}{(\tan(\psi))} \left[1 - \frac{3(l+h \tan(\psi))}{(l+2h)}\right] \left(1 - \frac{l}{2h} \frac{h}{(\tan(\psi))} \left(2l + h \tan(\psi)\right) + 3 \frac{3(l+h \tan(\psi))}{(l+2h)}\right)$</td>
<td>(31)</td>
</tr>
<tr>
<td>$\tau_s = \rho gh \sin(\psi) \frac{h}{(\tan(\psi))}$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Scheme III: $l &gt; L_{FC}$</th>
<th>External forces:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_w = W_n - q_n(l - L_{FC}) - F_t$</td>
<td>$T_w = W_h - k_f q_n(l - L_{FC})$</td>
</tr>
<tr>
<td>$M_w = (W_n - q_n(l - L_{FC})) \frac{L_{FC}}{2} + W_h \frac{h}{2} - F_t \left(L_{FC} + \frac{L_{FC}}{2}\right)$</td>
<td>(32)</td>
</tr>
<tr>
<td>Compressive $\sigma_c$ and shear $\tau_s$ stress:</td>
<td></td>
</tr>
<tr>
<td>$\sigma_c = \frac{3}{4} \rho gh \cos(\psi) \frac{L_{FC}}{(\tan(\psi))} \left(1 + \frac{1}{l+2h} \left[4h \frac{L_{FC}}{2} \tan(\psi) - (l+2L_{FC})\right]\right)$</td>
<td>(33)</td>
</tr>
<tr>
<td>$\tau_s = \rho gh \sin(\psi) \frac{h}{(\tan(\psi))} \left(1 - k_f \frac{L_{FC}}{(\tan(\psi))}\right)$</td>
<td></td>
</tr>
</tbody>
</table>
critical crack lengths $l_{cw}$ (in compression) and $l_{cs}$ (in shear). The critical crack length $l_{c}$ is the minimum of the latter two values. Note that $l_{c}$ is distinct from the critical crack length $l_{s}$ for the slab tensile failure.

4. Model Application

In this section, the features of the presented Propagation Saw test model are highlighted. First, a realistic study case is introduced, which is helpful to show how the model can predict the results of the PST. The main results of a parametric study are then presented, while the details are discussed further in the Appendix. Finally, the model is compared with experimental data.

4.1. Simple study case

A simple PST case study is presented to show the influence of the crack length on the stresses in the slab and in the weak layer, as well as to link the quantitative results (critical crack length and propagation length) to the modeled PST outcomes (END or SF). In this case study, the slab has a width of $b = 30$ cm and a height of $h = 30$ cm. The total length of the specimen is $l_{tot} = 2$ m and the slope angle is $35^\circ$. Finally, the thickness of the weak layer is $h_{w} = 1$ mm whereas the static friction coefficient between the rigid substratum and the slab is $k_f = 0.5$, equivalent to a friction angle of $\sim 27^\circ$ (van Herwijnen and Heierli, 2009).

The compressive strength in the weak layer is 2.5 kPa, while the shear strength is 0.5 kPa, similarly reported by Reiweger et al. (2015). For the snow in the slab, the elastic modulus $E$ is a function of the density, as in Scapozza (2004):

$$E(\rho) = 1.873 \cdot 10^5 \exp^{0.0140\rho}$$

and the tensile strength is given by Sigrist (2006):

$$\sigma_{yst} = 2.4 \times 10^5 \left(\frac{\rho}{\rho_{ice}}\right)^{2.44}$$

where $\rho_{ice} = 917$ kg/m$^3$. Recalling expressions in Table 1 and Table 2, it is possible to plot the tensile stress in the slab and the compressive and shear stresses in the weak
Figure 12 depicts the case for a slab density equal to $\rho = 230 \text{ kg/m}^3$. With regard to the tensile stresses in the slab, the initial part of the curve $\sigma_s^t$ shows a quadratic increase of stress due to the growing bending moment. Then, the beam tip comes into contact with the substratum and between $L_{IC}$ and $L_{FC}$ the stress locally reduces. For lengths greater than the full contact length $l > L_{FC}$, the bending moment is constant at the crack tip cross section but the stress increases linearly: at this point, the tensile force in the direction of the slope drives the experiment. In contrast, in the weak layer, the compressive $\sigma_w^c$ and shear stresses $\tau_w^s$ have a non-linear increase along the whole experiment and the effects of the contact of the slab with the substratum are less pronounced.

In Figure 12, failure, specifically when the stress reaches the maximum admissible stress, is indicated with the symbol “X”. As it was discussed previously, the slab fails under tensile stress (i.e. when $\sigma_s^t$ from Table 1 is equal to $\sigma_s^yt$), induced by the combined effect of bending and horizontal force. Similarly, the weak layer fails due to shear, as $\tau_w^s = \tau_w^s$, or due to compression, as $\sigma_w^c = \sigma_w^yc$, where $\tau_w^s$ and $\sigma_w^c$ are computed following Table 2. However, it is difficult to relate the various stresses distribution at failure.
due to the significant difference in the strength of each layer. Then, to have a better understanding of the phenomena in the Propagation Saw Test, the stresses $\sigma^w_{st}$, $\sigma^w_{sc}$, $\tau^w_{ys}$ are scaled with respect to their admissible stress values $\sigma_{styt}$, $\sigma_{scyc}$, $\tau_{ysys}$ as elaborated in the following paragraph.

Figure 13: Model application results. Figure (a), Full propagation (END) case: shear failure in the weak layer appears at 11 centimeters and propagates to the end of the specimen. Figure (b), Slab Fracture after propagation (SFa): the initial shear failure in the weak layer at 45 centimeters is followed by the slab fracture at 166 centimeters. Figure (c), Slab fracture before propagation (SFb): tensile failure in the slab happens at 59 centimeters, before weak layer failure, which would theoretically appear at 79 centimeters. Figure (d), Critical crack length for the slab fracture $l^c_{sc}$ and the weak layer failure $l^c_{ws}$. Note that $l^c_{ws}$ is the minimum between the critical crack length for shear $l^c_{ws}$ and compression $l^c_{wc}$ failure.
4.2. Full propagation (END) case

In this first case, the density of the slab is $\rho = 280 \text{kg/m}^3$, indicative of a stiff slab. Firstly, the elastic modulus $E(\rho)$ and the tensile threshold stress are:

$$E \left( \frac{280 \text{ kg/m}^3}{\text{m}^3} \right) = 12.15 \text{ MPa}$$

$$\sigma_{yt} \left( \frac{280 \text{ kg/m}^3}{\text{m}^3} \right) = 13.28 \text{ kPa}$$

Then, the characteristic lengths $L_{IC}$ and $L_{FC}$ are:

$$L_{IC} = 75 \text{ cm} \quad L_{FC} = \sqrt{3}L_{IC} = 131 \text{ cm}$$

The plot of normalized stresses is presented in Figure 13(a). On the basis of our hypotheses, the model predicts that the critical shear stress ratio in the weak layer reaches failure (i.e., unit value of $\tau_s/\tau_{ys}$), for a critical crack length of 11 centimeters. It is important to emphasize that, assuming a brittle failure criterion, failure due to shear (or compression) is instantaneous and the potential crack propagation not restrained unless the stress in the weak layer is suddenly reduced (e.g. due to slab fracture). Hence, the model suggests that crack propagation is initiated in the weak layer at $l = 11$ cm, which causes the stress in the slab to further increase. However, the tensile stress does not reach its maximum admissible value. Consequently, this result would represent the full propagation case (END).

It is possible to compute the propagation length by subtracting the critical crack length for the onset of crack propagation in the weak layer $l_w^c$ to the one required for slab fracture $l_s^c$. In the present case the crack would propagate to the end of the column and the propagation length would be:

$$l_p = l_s^c - l_w^c = 200 - 11 = 189 \text{ cm}$$

4.3. Slab Fracture after propagation (SFa) case

Consider a slab density of $\rho = 230 \text{ kg/m}^3$. The elastic modulus $E(\rho)$ and the tensile threshold stress are:

$$E \left( \frac{230 \text{ kg/m}^3}{\text{m}^3} \right) = 5.77 \text{ MPa}$$
\[ \sigma_{st} \left( 230 \text{ kg/m}^3 \right) = 8.22 \text{ kPa} \] (43)

and, consequently, the characteristic lengths of initial \( L_{IC} \) and of full contact \( L_{FC} \) are:

\[ L_{IC} = 66 \text{ cm} \quad L_{FC} = \sqrt{3} L_{IC} = 114 \text{ cm} \] (44)

In Figure 13(b), the evolution of the stresses is plotted. Shear is the cause of failure in the weak layer. To create the onset of crack propagation, a critical crack length of 45 centimeters is required. Then, the weak layer progressively collapses and, once the crack reaches a length of 166 centimeters, the increased tensile stress induces slab failure. This case would represent the slab fracture case after propagation (SFa).

The propagation length is computed by subtracting the critical crack length \( l_{w}^c \) from \( l_{s}^c \), and, in this case, it is:

\[ l_{p} = l_{s}^c - l_{w}^c = 121 \text{ cm} \] (45)

4.4. Slab Fracture before propagation (SFb) case

The last case study considers a slab with a density of \( \rho = 180 \text{ kg/m}^3 \), which, given the previous assumptions of elastic modulus and tensile strength as function of the density, results in a low value of \( E \) and \( \sigma_{st} \):

\[ E \left( 180 \text{ kg/m}^3 \right) = 2.74 \text{ MPa} \] (46)

\[ \sigma_{st} \left( 180 \text{ kg/m}^3 \right) = 4.52 \text{ kPa} \] (47)

The characteristic lengths of initial contact \( L_{IC} \) and of full contact \( L_{FC} \) are:

\[ L_{IC} = 58 \text{ cm} \quad L_{FC} = \sqrt{3} L_{IC} = 101 \text{ cm} \] (48)

In Figure 13(c), the ratio of tensile stress to tensile strength in the slab reaches the unit value, meaning that the slab fractures at 59 cm, before the weak layer can fail. This, in turn, causes the reduction of the stresses in both the weak layer and the slab. Figure 13(c) shows dashed lines for the shear \( \tau_{ws}^c \) and the compression \( \sigma_{wc}^c \) stresses after the slab fracture, since the test has already concluded. If sawing continued, the model suggests that the test would be equally repeated.
Finally, the propagation length, following the previous definition as the difference of the critical lengths, $l_c^s$ and $l_c^w$, is negative:

$$l_p = l_c^s - l_c^w = 59 - 79 = -20 \text{ cm} \quad (49)$$

as an indication of slab fracture occurring before crack propagation in the weak layer.

### 4.5. Propagation length with respect to density

Considering the signed propagation length with respect to the density variation of the slab, it is possible to characterize the Propagation Saw Test set-up and its outcomes. Firstly, density is considered to vary from 50 to 300 kg/m$^3$, being a typical range found in slab layers (Schweizer, 1999). For each density value, the critical length for the weak layer $l_c^w$ and for the slab $l_c^s$ is computed. In order to simplify the computation of propagation length, the Scheme II curve ($L_{IC} < l < L_{FC}$) is replaced with its linear interpolation, as pictured in Figure 12 (dashed lines). This is justified by the fact that for the weak layer the linear approximation is close to the actual solution, whereas in the slab, the stress at $l = L_{IC}$ is greater than most of the values in the middle range.

In Figure 13(d), the critical crack lengths, for weak layer failure and slab fracture, are plotted against density. Recall that a direct dependence of the elastic modulus and of the tensile strength from the slab density was assumed and, as a result, the problem depends on the geometrical properties of the isolated volume of snow, on the strength of the weak layer and on the slab density. The length of slab fracture $l_c^s$ is increasing with the density since the resistance of denser cohesive snow grows faster than the applied load. From 180 kg/m$^3$, the plot presents a change in the behavior, which corresponds to the effect of the load reduction due to the contact with the rigid substratum ($l > L_{IC}$).

For densities larger than 240 kg/m$^3$, the slab critical crack length $l_c^s$ for tensile failure is greater than $l_{tot}$ (dashed line) and the total specimen length should be considered instead.

In the weak layer, the critical length reduces as the load increases. Both compressive and shear failure give a monotonic descending curve and the critical length $l_c^w$ is given by the minimum of the two values $l_c^{w,c}$ and $l_c^{w,s}$. Compressive failure is predominant at the beginning, for very low values of $\rho$, whereas, for higher densities, shear
Figure 14: Signed propagation length with respect to the slab density for a 0.3 × 0.3 × 2 m³ specimen on a 35° slope. The different model results suggest full propagation (END) for densities in the range 249 to 300 kg/m³, slab fracture after propagation (SFa) for densities in the range 191 to 249 kg/m³ and slab fracture before propagation (SFb) for lower values of densities.

plays a major role. In the model, the critical crack length reaches zero at 300 kg/m³, which would be equal to weak layer failure without sawing (i.e. zero crack length).

Finally, the length of propagation is computed by subtracting \( l_{cw} \) from \( l_{sc} \). Figure 14 demonstrates the resulting plot. As a result, it is possible to identify the test outcome for each density value. For slabs with low densities, up to 191 kg/m³, the model predicts slab fracture before propagation (SFb) in the weak layer. Then, as the slab density increases, an increase of the elastic modulus and tensile strength in the slab was assumed. As the slab becomes stronger than the weak layer, for densities within the range 191 – 249 kg/m³, the computed result is slab fracture after propagation (SFa). Finally, the model shows full propagation (END) from 249 to 300 kg/m³, as the propagation length becomes bigger than the total length of the specimen (in this
example, 2 meters).

5. Parametric study

This section presents the main results from the parametric study. Further details and figures are provided in the Appendix.

Firstly, the effect on the model results is addressed, in terms of critical crack length and propagation length for a change in slab or weak layer strength. Figure 17(a) and (b) indicate that an increase in the weak layer strength leads to an increase in the critical crack length \(l_{wc}\) and \(l_{wc,s}\), as expected. In addition, Figure 17(c) shows that the critical crack length for slab fracture \(l_{s}^{c}\) is very sensitive to the tensile strength in the slab. More precisely, an increase of 10% in the tensile strength results in doubling the propagation distance.

Increasing the slope angle \(\psi\) causes the applied shear force in the weak layer to increase. In the slab, a competitive mechanism between bending moment and tensile force is present. As expected, the critical crack length (in shear \(l_{s}^{c}\)) decreases with increasing slope angle (Figure 19(c)). On the other hand, the propagation length \(l_{p}\) is larger for steeper slopes (Figure 19(d)).

Furthermore, the weak layer critical crack length reduces with larger values of slab height \(h\), since it represents an increase of the gravitational load (Figure 20(c) and (d)). Concerning the propagation distance, the effect of slab height is two-fold. In the cantilever regime, an increase in slab height leads to an increase of the propagation distance, while once the slab is in contact with the weak layer, the behavior changes due to the additional friction effects (Figure 20(a)). In that case, the propagation distance decreases with increasing slab height. This change of behavior occurs for densities higher than 200 kg/m\(^3\) (Figure 20(d)).

The variation of density, which in the model is linked to the elastic modulus and the tensile strength, has a similar effect as the variation of the slab height. An increase in density induces an increase in the load and, consequently, a decrease of the critical crack length \((l_{wc}^{p}\) and \(l_{wc,s}^{p}\)). For the slab, the increase in density leads to a linear increase of the load but also an exponential increase of the maximum admissible tensile stress,
which in turns induces higher values of propagation distance (Figure 21(a)).

In the proposed model, the thickness of the weak layer $h_w$ is only a geometrical parameter, which has no effect on the failure behavior of the weak layer in shear. However, a small decrease of $h_w$ yields a slight increase of the critical crack length in compression $l_{wc}$. In this case, the contact between the slab and the weak layer occurs for smaller crack lengths and, thanks to this, the tensile stresses in the slab are reduced. Furthermore, the weak layer thickness has an important effect on the propagation distance. The model suggest that thin weak layers promote full propagation, since the slab is in contact with the rigid substratum already for small crack length and the slab is mostly subjected to horizontal forces; instead, thick weak layers promote slab fracture, being the tensile stress increasing quadratically with the crack length and the problem is mainly driven by bending (Figure 22).

Surprisingly, the substratum static friction coefficient $k_f$ plays an important, yet complex role in the possibility of full propagation or slab fracture. The static friction, however, does not influence weak layer failure. Since friction reduces the load in the horizontal direction, the propagation length generally increases with a higher friction coefficient and its values can modify the test outcome for large slab densities ($\rho > 200 \text{ kg/m}^3$), for which SFa and END cases are expected. An increase in the friction coefficient promotes full propagation since it causes a reduction of the tensile stresses in the slab upon contact (Figure 23(d)).

6. Comparison with experimental data

In this section, the experimental values in Figure 13 of the work of Gaume et al. (2015) are reported and compared with the analytical prediction of the proposed model. The data in the cited paper is presented in Table 3 and consist of the slab critical crack length (to be compared with its analytical counterpart $l_c$), the number of experiments $N_{exp}$, the slab height and the slab density.

The PST results depend strictly on the slab height which, in turn, is subjected to consolidation. In this comparison, in order to correctly take into account different slab thicknesses found in-situ, the dependence of the density with respect to the height of
the slab is considered. A simplified linear dependence is extracted from the data in Table 3. For a slab of height varying from 30 cm to 100 cm, the density increases almost linearly from 100 kg/m$^3$ to 350 kg/m$^3$. As it was discussed before, the elastic modulus and the tensile strength are assumed to be function of density, according to equations (36) and (37).

<table>
<thead>
<tr>
<th>$\rho$ (kg/m$^3$)</th>
<th>50-100</th>
<th>100-150</th>
<th>150-200</th>
<th>200-250</th>
<th>250-300</th>
<th>300-350</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_{exp}$</td>
<td>1</td>
<td>23</td>
<td>37</td>
<td>32</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>$&lt; h &gt;$ (m)</td>
<td>0.3</td>
<td>0.4</td>
<td>0.5</td>
<td>0.65</td>
<td>0.8</td>
<td>1.0</td>
</tr>
<tr>
<td>$M (l_s^c/l_{tot})$</td>
<td>0.26</td>
<td>0.42</td>
<td>0.43</td>
<td>0.50</td>
<td>0.69</td>
<td>0.85</td>
</tr>
<tr>
<td>$P_{25} (l_s^c/l_{tot})$</td>
<td>-</td>
<td>0.23</td>
<td>0.26</td>
<td>0.31</td>
<td>0.52</td>
<td>0.8</td>
</tr>
<tr>
<td>$P_{75} (l_s^c/l_{tot})$</td>
<td>-</td>
<td>0.45</td>
<td>0.50</td>
<td>0.69</td>
<td>0.91</td>
<td>0.91</td>
</tr>
</tbody>
</table>

Table 3: PST field data from Gaume et al. (2015). The number of experiment $N_{exp}$, the average slab height $< h >$, the median $M$, the 25th percentile $P_{25}$ and the 75th percentile $P_{75}$ of the ratio between slab critical crack length $l_s^c$ and total specimen length $l_{tot}$ are given with respect to the slab densities.

Figure 15 depicts the proposed comparison between the experimental data (represented by the box plot) and the model prediction (shown with the three line plots). The solid blue line is the value of the critical crack length $l_s^c$ given by the model when the tensile strength $\sigma_{yt}$ is given exactly by expression (37). The two dashed lines represent the value of $l_s^c$ for a $\pm 5\%$ variation in $\sigma_{yt}$.

The overall trend of increasing propagation length with respect to density is captured by the model. Indeed, the goodness fit indicator computed between the analytical results and the experimental data is $R^2 = 0.912$, confirming the relevance of the increasing trend.

In the range of densities between 50 and 200 kg/m$^3$, the analytical solution is almost linear, with a very small change for the $\pm 5\%$ strength variation. In this interval, the model predicts a slightly lower critical crack length for slab fracture than the reported data. For densities higher than 200 kg/m$^3$, the increase of $l_s^c$ is observed in both the experimental and analytical values. The model predicts a sudden increase of $l_s^c$ after 200 kg/m$^3$. The variation of results provided by the model for the relatively small
change in tensile strength is consistent with the field data. In particular, the model suggests with the values of $l_s$ that large densities tend to promote full propagation.

Figure 15: Comparison of the model prediction and the experimental critical crack length values with respect to snow density for the case of slab tensile fracture in the PST. The solid blue line is the computed analytical value $l_s$. The two dashed lines represents the model predictions with a variation of $+5\%$ and $-5\%$ of the tensile strength in the slab. The statistics of the experimental results are presented using the box plot. The green boxes mark the 25 and 75 percentile whereas the red line is the median of the field data.

7. Discussion

7.1. Results from the study and comparison with previous works

In this work, we developed a new analytical model of the Propagation Saw Test (PST) based on the Euler-Bernoulli beam theory. The proposed model links the critical crack length and the propagation distance to the two main results of the PST, namely full propagation (END) and slab fracture (SF).
Given the assumption of increasing elastic modulus and tensile strength of the slab with increasing density (Scapozza, 2004; Sigrist, 2006), the increasing trend of propagation distance with density observed in field experiments (Gaume et al., 2015) was also recovered by our model. In addition, for low density slabs (below \( \sim 190 \text{ kg/m}^3 \)), slab fracture is predicted to occur before reaching the weak layer critical crack length (SFb); for intermediate densities (\( \sim 190 - 250 \text{ kg/m}^3 \)), the model suggests the occurrence of slab fracture after the onset of crack propagation in the weak layer (SFa); finally, large densities (above \( \sim 250 \text{ kg/m}^3 \)) appears to lead to full propagation in the weak layer without slab fracture (END). This result is in line with the findings of Gaume et al. (2015) which reported, using the discrete element model, full propagation for densities larger than \( \sim 280 \text{ kg/m}^3 \).

It is important to highlight that the mechanical quantities are introduced only in the description of the stresses in the weak layer and the slab through the values of \( L_{IC} \) and \( L_{FC} \). In particular, the length of initial contact \( L_{IC} \) is a key parameter since it marks the change of stress regime in the slab, limiting the increase of the tensile stresses. Therefore, the contact between the slab and the substratum is a fundamental aspect of the Propagation Saw Test, being responsible of promoting full propagation (END). In contrast, the weak layer is less affected by this change in force schemes, being mainly dependent on the inverse of the resisting cross section \( b (l_{tot} - l) \).

The proposed model considers the possibility of having compressive or shear failure in the weak layer and tensile failure in the slab. The parametric study revealed that the critical crack length in the weak layer \( l_{w}^c \) decreased with increasing depth and density of the slab (when slab elastic modulus and tensile strength depend on the slab density itself) and with decreasing weak layer strength.

In addition, it was shown that the critical crack length \( l_{w}^c \) was mostly driven by the shear failure (except for flat terrains) and it was smaller for larger slope angles. This is in line with the recent results of Gaume et al. (2017) and contradictory to the anticrack theory (Heierli et al., 2008), although the weak layer collapse was accounted for. This trend was also recently suggested by van Herwijnen et al. (2016) who showed using FEM simulations that the mechanical energy provided by the anticrack model was too low for steep slopes.
Concerning the slab fracture, the sensitivity analysis showed that the propagation distance was significantly influenced by the tensile strength of the slab, where the higher the strength, the longer the propagation distance, as expected. It was demonstrated that the model is sensitive to the tensile strength value in the computation of propagation distance, as a large variation of results were observed for a relative small change of ±5%. Furthermore, the propagation distance increased with increasing slope angle and with decreasing weak layer thickness, as shown also by Gaume et al. (2015) using the discrete element method.

7.2. Limitations of the mechanical model

Although our model was able to compute critical crack lengths and propagation lengths indicating a relationship with possible slab fracture (SF) and full propagation (END) cases, clearly the method is not capable to address certain test results, such as “self arrest” ARR or “en-echelon fractures” (van Herwijnen and Jamieson, 2005; van Herwijnen et al., 2010; Gauthier and Jamieson, 2010), since it is based on deterministic assumptions. The present model assumes homogeneous snow in the isolated volume and, consequently, it is related to the average mechanical characteristics of each snowpack layer.

Moreover, a single homogeneous slab was considered instead of a multilayered continuum. However, the proposed approach may still be used in the case of multilayered slab by computing its bulk elastic modulus, using either the finite element method (FEM) (Reuter et al., 2015) or the analytical model developed by Monti et al. (2016).

The presented model considers a rigid interface between slab and weak layer and the use of the Euler-Bernoulli theory implies that such interface rotates uniformly under bending moment, i.e. the resisting weak layer is uniformly loaded. This assumption was introduced in order to independently analyze the stress evolution using standard Euler-Bernoulli beam theory and taking advantage of the superposition of effects. Additionally, it allows to consider the weight of the slab to be concentrated in its center of gravity in the computation of the stresses in the weak layer while considering perfect boundary conditions in the slab. Although this approach results in an easier analytical description of the model, it does not take into account important effects due to defor-
mation. By considering the interface between the slab and the weak layer as rigid, a perfect transfer of the load is enforced. Moreover, this hypothesis implies an infinite distribution of stresses for an infinite specimen length $l_{tot}$, which has also been proven to be untrue in field measurements. Indeed, it was shown that stresses redistribute over a finite length related to elastic properties of the slab and the weak layer (Chiaia et al., 2008; Gaume et al., 2013; van Herwijnen et al., 2016; Gaume et al., 2017).

The presented framework is quasi-static, with a pure brittle failure criterion for both the weak layer and slab. As discussed by Sigrist and Schweizer (2007) and later by Schweizer et al. (2011), fracture energy can be measured and has a key role in the PST. The use of a linear elastic model overestimates the elastic strain energy, resulting in a lower value of critical crack length. Instead, by considering the energy dissipation during fracture, the critical crack lengths would very likely be larger. Furthermore, the model does not take into account the dynamic effects observed during crack propagation in the PST. From a theoretical point of view, the lack of dynamic effects implies that the stresses predicted in the slab are higher than expected (e.g. see Gaume et al. (2015)), and that larger values of propagation length $l_p$ are found.

In addition, the failure criterion used is a simplified one in which the single stress components are compared directly with their threshold value. This criterion is a simplification of the mixed-mode Mohr-Coulomb failure surface presented by Reiweger et al. (2015), shown in Figure 16. However, the simplified failure criterion contains the main components of shear and compressive failure, given similar strength values. The presented framework can be extended to many more precise and complex failure criteria.

8. Conclusions

A new mechanical model of the Propagation Saw Test (PST) was developed based on the Euler-Bernoulli beam theory using a purely brittle failure criterion for the slab and the weak layer. The model allowed to compute the stress states in the slab and in the weak layer as a function of the crack length, geometrical and mechanical parameters. By comparing the computed stress field with their respective strengths, it was possible
to compute the critical crack length for weak layer failure and slab fracture and the propagation distance.

The presented model shows a significant agreement between the quantitative computations (critical crack lengths and propagation distance) and the two main outcomes of the PST, namely full propagation (END) and slab fracture (SF). Assuming a relationship between slab tensile strength, slab elastic modulus and slab density, it is possible to have a quantitative assessment of the PST using geometrical properties, weak layer strength and the slab density itself. In such case, for low density slabs (and thus low elastic modulus and tensile strength), slab fracture occurs before reaching the weak layer critical crack length (SFb); for intermediate densities, slab fracture is predicted after the onset of crack propagation in the weak layer (SFa); large densities (i.e. large elastic modulus and tensile strength) suggested a full propagation in the weak layer without slab fracture (END). Furthermore, for realistic snowpack properties, the overall trend of increasing propagation distance with density observed in field experiment was well reproduced by the proposed model. In addition, the model showed a dependence of the critical crack length for the onset of crack propagation in the weak layer from the slope angle, with decreasing values of $l_{cr}$ on steeper slopes.

The introduced model is novel approach to interpret the PST field experiments, by considering explicitly the complex interplay between crack propagation in the weak layer.
layer and the slab fracture. Nevertheless, future work is required to develop a better mechanical coupling between the slab and the weak layer with particular focus on the deformation effects and the connected redistribution of stresses at the layers interface. This could lead to the introduction of a characteristic length that describes stress concentrations close to the crack tip.

Moreover, further model validation is essential to test thoroughly the proposed approach. In order to strengthen a possible connection between the model predictions and the results from field PST, a comparison with a large dataset of detailed in-situ experiments is required. In practice, this would imply a better characterization of the failure properties of the weak layer in the field (including its failure envelope), which is required as input in the model. A reliable experimental validation would allow to point up possible shortcomings and improve the analytical approach.

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Appendix A Parameters sensitivity analysis

The objective of this section is to illustrate the dependence of the results on the variables of the model. The following section is a key point in order to determine the relative importance of each of the numerous parameters. The base experiment in analysis is the same as the previous sections with a slab of section $30 \times 30 \text{cm}^2$ and a total length of $l_{\text{tot}} = 2 \text{ m}$. Where unspecified, the slab has a density of $230 \text{ kg/m}^3$. The weak layer thickness is $h_w = 1 \text{ mm}$ and the static friction coefficient between the substratum and the slab is $k_f = 0.5$. Finally, the slope angle is $35^\circ$. In every subsection, each parameter of the model is studied in order to evaluate the variation of critical crack length and propagation length.

A.1 Strength values

In order to test the sensitivity of the results to a variation in the threshold stress, the strength was modified by $\pm 5\%$ and $\pm 10\%$.

Firstly, the weak layer strength perturbation is addressed. The resulting critical length for the compressive and shear cases are presented in Figure 17(a) and 17(b) respectively. In both plots, a moderate variation in the critical crack length was observed with respect to the change in strength. In the first case, the $10\%$ compressive strength reduction causes a maximum change of critical crack length of about 10 centimeters, corresponding to a $5\%$ of the total specimen length (2 meters). Similarly, in the shear strength case, the maximum change is given by the $10\%$ reduction, which gives about 38 centimeters less in critical crack length, corresponding to a $19\%$ of the total specimen size (2 meters).

Regarding the slab, a similar perturbation of the strength is considered, varying it $\pm 5\%$ and $\pm 10\%$. The results are presented in Figure 17(c). Clearly, the major effect induced by a small variation in the slab tensile strength. A $10\%$ increase caused up to doubling of the slab fracture length (405 centimeters, corresponding to a $202\%$ difference), whereas a $10\%$ reduction provided a maximum decrease of $104\%$, corresponding to about 208 centimeters. Notably, these values are computed focusing on the analytical result and disregarding the fact that, even with a total specimen length is 2 meters, the model provides unrealistic values.
Figure 17: Sensibility analysis of critical crack lengths with respect to a variation in (a) weak layer compressive strength $\Delta \sigma_{yc}$, (b) weak layer shear strength $\Delta \tau_{ys}$ and (c) slab tensile strength $\Delta \sigma_{yt}$. 
Figure 18: Propagation length with respect to density resulting from the sensibility analysis in the case of variation of (a) weak layer strengths $\sigma_{w}^y$, $\tau_{w}^x$, and (b) slab strength $\sigma_{s}^y$. It appears that, for the presented model, the change in weak layer resistance has a moderate effect. Vice versa, a small perturbation in the slab tensile strength provokes a large variation in the outcomes of the test in the $150-300$ kg/m$^3$ density range. In particular, the propagation length approximately doubles with a 10% change of $\sigma_{s}^y$. 
The changes in critical crack lengths were used to compute the signed propagation length. Figure 18(a) shows the envelope solution due to the perturbation of compressive and shear strengths in the weak layer. The moderate deviation from the original case is visible, with a maximum variation of propagation length of about 20 centimeters (10% of the total specimen length). Further, Figure 18(b) depicts the substantial change in the propagation length due to slab strength variation.

This case not only provides substantially different values of propagation length, highlighting the considerable sensitivity of the PST to $\sigma_{st}$, but also shows how a 1% $\sim$ 2% reduction of the tensile strength implies a slab fracture (SFa) instead of full propagation (END). This emphasizes that snowpack variability can substantially influence the PST outcome leading to a difficult interpretation of results.

This is a fundamental aspect of the Propagation Saw Test. The current analysis proves the tight dependence of the outcomes from the tensile strength $\sigma_{st}$, highlighting that the PST has been developed in order to evaluate, not only the crack propagation propensity in weak layers, but also slab fracture.

A.2 Slope angle

The slope angle is a key factor in avalanche terrain. Its influence on the critical crack and propagation length from the flat cases to very steep inclines is investigated. The value of $\psi$ ranges from 0° to 60°. In Figure 19(a), the critical ratio for the tensile stress in the slab is presented. Firstly, the value of the length of initial contact $L_{IC}$ and the length of full contact $L_{FC}$ are not constant. The steeper the slope, the longer cantilever is required to have contact between the slab and the substratum. Eventually, $L_{IC}$ and $L_{FC}$ tend to infinity for a 90° inclination, since less vertical load will be applied to the slab for increasingly higher angles.

The initial contact length represents the change in the regime of stresses in the slab from the cantilever (I) to the hinged (II) and to the fixed roller (III) schemes. If there was no change in the stress variability during the PST, then there would not be the possibility to have full propagation cases (END), since the quadratic increase of stress would limit the critical length of failure for the slab.

The tensile stress in the slab is the result of the sum of bending and normal force.
Figure 19: Model dependence to slope angles $\psi$ (from $0^\circ$ to $60^\circ$ with $10^\circ$ steps): Figure (a), (b) and (c) show respectively the tensile stress in the slab, the compressive and the shear stresses in the weak layer plotted against crack length; Figure (d) depicts the propagation length with respect to the slab density.

- The compressive stress in the weak layer is slightly variable along the proposed range of slope angles, as observed in Figure 19(b). All the critical lengths for this case are concentrated around 50 centimeters and the horizontal case does not represent the worst case scenario. Contrary to this, the shear stress strictly depends on the slope angle, as depicted in Figure 19(c). In particular, the critical crack length significantly decreases with increasing slope angle. By comparison of the two plots, it is possible to
suggest that for a slab density of 230 kg/m³, compressive failure is dominant up to 25° whereas shear stress is the cause of failure for steeper slopes. In the case of 10° angle, it is also possible to see that the friction of the rigid substratum absorbs all applied shear load.

In Figure 19(d), the propagation length is plotted against the density for various slope angles. Firstly, the propagation length increases with increasing slope angle. Secondly, at the selected density of 230 kg/m³, the slab fracture after propagation (SFa) is predicted for the range of angles from 0° to 30°. In particular, the cases of flat, 10° and 20° present approximately the same propagation length. The case for 30° is on the limit of the full propagation case, since the slab tensile stress provokes failure close to the total length of the specimen. Finally, the model predicts full propagation (END case) for high values of the slope angle, i.e. $\psi \geq 40°$.

A.3 Slab depth

The effect of the variation of the slab depth on the Propagation Saw Test is analyzed. The load on the specimen is proportional to the height $h$. However, the moment of inertia of the snow beam increases with the cube of the height. Consequently, the tensile stresses $\sigma^t_s$ are limited for very thick slabs, as seen in Figure 20(a). At the same time, the weak layer is subjected a proportional increase in stress, as visible in Figure 20(b) for the compressive part and in Figure 20(c) for the shear one.

Finally, Figure 20(d) depicts the propagation length against the density values, with respect to different slab heights. For low density of the slab, the propagation length increases proportionally to the value of $h$. On the contrary, starting from about 180 – 200 kg/m³, an inversion of the trend is observed and a smaller slab has a larger length of propagation. It is also possible to observe an asymptotic behavior, with a limit line in the plot for both the lowest and highest values of $h$.

On the lower end, the asymptote is given by the fact that the plot is approaching the case of zero thickness slab, i.e., zero load as well as zero resistance are predicted. The strength depends on the cube of the height of the slab whereas the bending and tensile load depends on its square at most. Hence, the predicted stress could grow to infinity.

But the maximum threshold for the tensile stress limits such value, so the propagation
Figure 20: Model dependence to slab depth $h$ (from 10 cm to 60 cm with 10 cm steps): Figure (a), (b) and (c) show respectively the tensile stress in the slab, the compressive and the shear stresses in the weak layer plotted against crack length; Figure (d) depicts the propagation length with respect to the slab density.

On the higher end of the plot, the solution converges for very thick slabs. Its high stiffness results in very low values of deflection of the tip. At the same time, the larger volume of snow implies a sustained load on both layers. In this case, the value of $L_{IC}$ is large and the stress evolution is mainly given by the Scheme I. The quadratic increase in tensile stress limits the critical length in the slab. The effect of the friction angle $k_f$ is increasingly less important as the slab is thicker. As shown in Figure 20(d), this occurs between the 45-50 centimeters and 55-60 centimeters, since no contact with the rigid substratum is predicted before slab fracture. This explains the asymptotic value for high values of $h$. 


Figure 21: Model dependence to density $\rho$ (from 50 kg/m$^3$ to 300 kg/m$^3$ with 50 kg/m$^3$ steps): Figure (a), (b) and (c) show respectively the tensile stress in the slab, the compressive and the shear stresses in the weak layer plotted against crack length.
A.4 Density

The propagation length plots are in Figure 13(d) and Figure 14 with the analysis of the density variation. With increasing values of density, the slab increases its weight and its tensile strength. Figure 21(a) shows this competitive mechanism. In the same plot, the curves for the initial and full contact length are superposed. For low values of $\rho$, the tensile strength is not enough to support the stresses applied by the cantilever beam, which then results in small critical crack length for slab fracture. As previously observed, for intermediate values of density, the strength increases to allow suddenly larger values of crack lengths. Further increase of density causes a substantial load growth and a reduction of $l'_C$. Finally, Figure 21(b) and Figure 21(c) depict the variation of compressive and shear stress in the weak layer with respect to density in the slab. In this case, $\rho$ contributes only to the applied load in the weak layer and this fact is clearly visible in the plot.

A.5 Weak layer thickness

The thickness $h_w$ is the only geometrical parameter of the weak layer. This value affects the initial contact $L_{IC}$ and the full contact length $L_{FC}$. Consequently, it has an effect in discriminating Scheme I, II and III. In Figure 22(a), the slab tensile stress for the $h_w$ variation is presented. As it was expected, the result is a family of parallel shifted curves. All the cases coincide in the initial part, corresponding to the cantilever scheme. If the weak layer is thin enough (in this case up to 1 millimeter), the force scheme changes due to contact with the substratum before slab fracture takes place. Otherwise, for thicker weak layers, the critical length is fixed and independent from $h_w$. In such cases, the values for the slab and weak layer critical crack lengths are unique.

In Figure 22(b) and Figure 22(c) the compressive and shear stress evolution are depicted. Clearly, the dependence of the stresses on the thickness of the weak layer is marginal.

Figure 22(d) shows the propagation length plotted against the density for various values of $h_w$. As previously stated, for small values of weak layer thickness, contact between the slab and the substratum is predicted. As a consequence, the propagation
Figure 22: Model dependence to weak layer thickness $h_w$ (from 0.5 mm to 2.5 mm with 0.25 mm steps): Figure (a), (b) and (c) show respectively the tensile stress in the slab, the compressive and the shear stresses in the weak layer plotted against crack length; Figure (d) depicts the propagation length with respect to the slab density.

- The propagation length is large and it decreases with increasing values of $h_w$. This results in a larger range of densities for which it is possible to have full propagation (END) in the PST.
- On the other hand, for thicker weak layers, the propagation length plot converges to a single curve, predicting only the case of slab fracture (SFa and SFb).

#### A.6 Friction

In Figure 23(a), Figure 23(b) and Figure 23(c), the tensile stress in the slab and the compressive and shear stresses in the weak layer are presented. While in the weak layer the effect of the friction coefficient $k_f$ is small, in the slab, the change is visible. Recall that the friction between the substratum and the slab is engaged only for the Scheme
Figure 23: Model dependence to substratum friction $k_f$ (from 0.0 to 0.5 with 0.1 steps): Figure (a), (b) and (c) show respectively the tensile stress in the slab, the compressive and the shear stresses in the weak layer plotted against crack length; Figure (d) depicts the propagation length with respect to the slab density.

III when the crack length is bigger than the full contact $L_{FC}$. In such case, the weight of the slab creates an additional friction force that counterbalance the tensile force in the slope direction.

In Figure 23(d), the propagation length with respect to the density is pictured in the case of friction coefficients $k_f$ varying from 0 to 0.5. Firstly, the effect of this parameter affects only a particular part of the plot. Indeed, this portion is the one that is correspondent with the Scheme III slab fracture.

As counterintuitive as it might be, having higher resisting forces due to friction could represent the worst case scenario, in terms of avalanche size. In fact, in the case of null $k_f$, the slab has to sustain its own gravitational load. When friction is present,
Figure 24: Comparison of propagation length against density in the case of null and $k_f = 0.5$ friction coefficients.

the total force is reduced instead. Consequently, this implies that, for the same tensile stress threshold, the diminishing friction case ($k_f = 0$) results in a slab fracture after propagation, whereas the high friction case ($k_f = 0.5$) turns out to be a full propagation case. The comparison between these two cases is shown in Figure 24, where the plot in Figure 14 is compared with the one predicted for the null friction case.

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