Parameterization of the spatially averaged sky view factor in complex topography

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Abstract

Large-scale meteorological, land surface or climate models require precise subgrid parameterizations to reproduce unresolved subgrid processes. This is particularly important in mountainous terrain where subgrid surface properties (e.g., glacier area extent) are used to illustrate the impact of global climate change. The sky view factor is one of the most frequently used terrain characteristics to parameterize the influence of topography on the surface radiation balance in complex terrain. A precise subgrid parameterization of sky view factors is therefore essential for accurate parameterizations of radiative fluxes such as shortwave diffuse sky and terrain reflected radiation. While its computation is straightforward for a particular site, a systematic characterization of the spatially averaged sky view factor is still missing. We use Gaussian random fields with a Gaussian covariance as topography model, and based on an analytical approximation, we derive a parameterization for the spatially averaged sky view factor. The sky view factor parameterization is solely based on computationally cheap terrain parameters, namely the correlation length of subgrid topographic features and the mean-squared slope in the grid cell of the large-scale model. For validation we compare parameterized to numerically exact sky view factors computed for real terrain of the Swiss Alps from a 25 m digital elevation model, with subsequent averaging to varying target domain sizes. We obtain a very good agreement when approaching the resolutions of regional climate models. The parameterization also compares well for various real topographies from the U.S.

1. Introduction

With about 27% of the Earth's land area covered by mountains, accurately modeling the radiation balance over topography is of great importance for large-scale models such as meteorological, hydrological, and climate models as well as for the topographic correction of satellite signals with coarse scale swath angles. Over the past decades, the horizontal spatial resolutions of climate and meteorological models have greatly improved, see, e.g., Dutra et al. (2011) for the European Centre for Medium-Range Weather Forecasts model or www.cosmomodel.org for the Consortium For Small Scale Modeling models. With decreasing grid sizes the incorporation of small-scale processes influenced by subgrid topography is crucial to increase model accuracy in large-scale models such as regional climate models [e.g., Leung and Ghan, 1995; Parajka et al., 2010; Dutra et al., 2011]. Furthermore, changes in small-scale surface properties in mountainous areas (e.g., glacial area or snow coverage) are frequently used to illustrate the impact of global climate change. The efficiency of subgrid parameterizations thus becomes increasingly important to improve model accuracy without computational overhead [Intergovernmental Panel on Climate Change, 2007].

Physically based subgrid parameterizations of radiative fluxes in large flat grid cells are necessary to account for unresolved topographic influences. Nowadays, some meteorological models account for subgrid topographic influences on the surface radiation balance by preprocessing the small-scale topography [e.g., Müller and Scherer, 2005; Senkova et al., 2007; Lai et al., 2010]. A few studies even proposed parameterizations of solar fluxes under cloudless sky [Dubayah et al., 1990; Essery and Marks, 2007; Lee et al., 2011; Helbig and Löwe, 2012]. Most of these parameterizations require precomputed subgrid sky view factors. Here we emphasize that on resolved topography the sky view factor \( F_{\text{sky}} \) is applied to describe limited sky view and terrain reflected/ emitted radiation in the shortwave (SW) and longwave (LW) radiation range. Variations in the sky view factor therefore relate to spatially variable incident radiation. The sky view factor \( F_{\text{sky}} \) thus also constitutes a key quantity in subgrid radiation parameterization schemes where it is frequently used to scale the same subgrid influences of topography on the surface radiation balance in large flat grid cells [e.g., Müller and Scherer, 2005; Senkova et al., 2007; Lai et al., 2010; Essery and Marks, 2007; Lee et al., 2011; Helbig and Löwe, 2012; Löwe and Helbig, 2012]. As an example, the SW diffuse sky radiation in a large flat grid cell...
with subgrid topography is parameterized by the product of the domain-averaged sky view factor and the diffuse sky radiation on a horizontal unobstructed grid cell. To correctly derive domain-averaged sky view factors a computational expensive preprocessing of topography is required to extract information on slope angles and horizon lines [Helbig et al., 2009]. A commonly applied simplification are discretized sky view factors where the horizon angle is computed for a finite number of azimuth directions around a surface [e.g., Corripio, 2003; Hock and Holmgren, 2005; Arnold et al., 2006; Lai et al., 2010; Manners et al., 2012]. However, it is not only intricate to accurately recover the actual horizon line in complex terrain by using fixed azimuth intervals, also horizon angles should actually be determined in a sloped coordinate system [Helbig et al., 2009]. A subgrid parameterization based on accurately computed sky view factors provides a solution to both and might therefore stimulate the application of a subgrid parameterization scheme for the SW and LW radiation balance in large-scale meteorological, land surface, and climate models. Also other applications such as the correction of satellite imagery from coarse-scale swath angles over mountainous terrain [e.g., Wen et al., 2009] involve similar quantities and might benefit from such a parameterization.

A simple power law relationship between $F_{sk}$ and standard deviation of slope angles was previously derived by Essery and Marks [2007] based on statistics from four real topographies from the U.S. A comparison of this expression with Gaussian random fields (GRF), however, indicated its limitation when applied to a broad range of terrain characteristics [see Helbig and Löwe, 2012]. Most recently, Löwe and Helbig [2012] derived an expression for the domain-averaged $F_{sk}$ by relating the joint probability density for the horizon and slopes to a level-crossing problem of the underlying topography. The resulting integrals were evaluated using an approximation for the level-crossing problem for GRF’s which was originally derived for the scattering of sound waves at sea surfaces for GRF’s [Wagner, 1967]. Practical fit formulas were derived for a broad range of domain-averaged slopes angles from about 7° to 60°, resulting in a subgrid parameterization for sky view factors which solely depends on the mean-squared slope [Löwe and Helbig, 2012].

The derivation was, however, based on a large ratio of the grid or model size $L$ to the typical width of the underlying topographic features $\xi$, i.e., on the limit $L/\xi \to \infty$. On the theoretical side, extending the parameterization developed by Löwe and Helbig [2012] by $L/\xi$ as a second scaling parameter origins from a dimensional analysis. GRF as topography model reduces topography to only two relevant length scales characterizing typical heights $\sigma$ and widths $\xi$ of topographic features in a domain size $L$. Practically, a derivation based on the limit $L/\xi \to \infty$ gives rise to uncertainties when applied to large-scale model with fixed grid size $L$. Due to the variability of the correlation length $\xi$ from cell to cell, different values of the “goodness” criterion $L/\xi \gg 1$ are obtained and systematic corrections can be expected. This has previously been shown for multiple terrain reflections on GRF [Helbig et al., 2009]. Therefore, a second scaling parameter, besides the mean-squared slope, is required for a reliable subgrid parameterization of $F_{sk}$.

In this article, we derive a subgrid parameterization for the sky view factor which does not require extensive preprocessing of the subgrid topography to compute nonlocal horizon angles. The parameterization can be applied in a subgrid radiation parameterization scheme over complex topography such as the one for SW radiation presented by Helbig and Löwe [2012] and Löwe and Helbig [2012]. The new subgrid parameterization for the sky view factor explicitly includes $L/\xi$ as a parameter. Thereby grid size dependent, systematic errors are minimized which enables wider application, e.g., in radiation schemes of large-scale models at different grid cell sizes. The parameterization is developed for GRF with a Gaussian covariance as a simplified topography model. We employ numerical methods which were previously derived within the radiosity approach for SW radiation balance [Helbig et al., 2009, 2010] to compute sky view factors exactly for a large ensemble of GRF. The obtained subgrid parameterization solely requires computationally cheap terrain parameters and consistently generalizes the analytical approximation of Löwe and Helbig [2012]. Despite apparent differences between the morphology of real terrain and GRF, the parameterization shows very good agreement with the sky view factors from the entire Swiss Alps computed from a 25 m digital elevation model and averaged to various target domain sizes. Parameterized and numerically exact computed sky view factors also agreed well for several, diverse real topographies from the U.S. We show that various details of real topography become irrelevant when parameterizing the sky view factor at horizontal resolutions of high-resolution regional climate models or land surface models of about 5 km. Our results demonstrate that simple, stochastic topography models such as GRF are equally well suited to derive accurate parameterizations.
2. Method

2.1. Gaussian Random Fields as Topography Model

Gaussian models are widely used to characterize random processes, which is largely because of its simple mathematical properties, and Gaussian random fields (GRF) are probably the best investigated type of random field (for an introduction, see Adler [1981]). Motivated by previous studies, where smooth GRF as stochastic topography model allowed us to systematically investigate radiative transfer in complex terrain via the radiosity approach [see Helbig et al., 2009; Helbig and Löwe, 2012; Löwe and Helbig, 2012], we use GRF here to systematically analyze domain-averaged sky view factors in complex terrain. Note that various relevant geometrical characteristics of real, complex topography can be reasonably well approximated by Gaussian statistics [Helbig and Löwe, 2012]. We use stationary, isotropic GRF's for terrain elevations \( z(\mathbf{x}) \) with a homogeneous mean elevation \( \bar{z} \). The following analysis is independent of the mean elevation which can be set to zero. We use a Gaussian covariance

\[
C(|\mathbf{r}|) = (z(\mathbf{x}) - \bar{z})(z(\mathbf{x} + \mathbf{r}) - \bar{z}) = \sigma^2 \exp\left[-\left(|\mathbf{r}|/\xi\right)^2\right].
\]

which solely depends on the magnitude of the lag vector \( \mathbf{r} \). GRF with covariance as in equation (1) have the advantage that topography is reduced to only two relevant length scales in a model domain \( L \): a valley-to-peak elevation difference \( \sigma \) (typical height of topographic features), which is the square root of the variance of the random field, and a lateral extension \( \xi \) (typical width of topographic features), which is the correlation length of the random field. We note that the restriction to isotropic GRF is motivated by a previous result where the standard deviations of slope components in orthogonal directions were roughly isotropic for a set of real topographies from the U.S. and Switzerland [Helbig and Löwe, 2012].

The overbar in equation (1) denotes ensemble averaging. Slope angles \( \zeta \) were computed from partial derivatives (slope components) \( \partial_x z \) and \( \partial_y z \) in orthogonal directions via \( \tan^2 \zeta = \left( \partial_x z \right)^2 + \left( \partial_y z \right)^2 \). Löwe and Helbig [2012] outline that the covariance in equation (1) implies a joint (slope) probability density\footnote{Löwe and Helbig, 2012} \( p_\zeta(\partial_x z, \partial_y z) = (2\pi\mu^2)^{-1} \exp \left\{ -\left(\left(\partial_x z\right)^2 + \left(\partial_y z\right)^2\right)/2\mu^2 \right\} \) which factorizes into two Gaussians with standard deviation \( \mu = \sqrt{2\sigma/\xi} \). As outlined by Löwe and Helbig [2012] the terrain parameter \( \mu \) is related to the mean-squared slope and can be computed from

\[
\mu = \left\{ \left(\left(\partial_x z\right)^2 + \left(\partial_y z\right)^2\right)/2 \right\}^{1/2}.
\]

using \( 2\mu^2 = \left(\partial_x z\right)^2 + \left(\partial_y z\right)^2 = \tan^2 \zeta = 4(\sigma/\xi)^2 \) which follows from the definition of the covariance given above [Adler, 1981]. The domain-averaged slope angle \( \zeta \) angle only depends on the ratio of \( \sigma \) to \( \xi \). By varying \( \sigma \) and \( \xi \) this allows us to create topographies with a variety of terrain characteristics for the same mean slope.

2.2. Characteristics of Model Topographies

We generated an ensemble of model topographies, i.e., digital elevation models (DEM) with discrete terrain elevations \( z_i \) from a multivariate Gaussian distribution with prescribed covariance (equation (1)). We chose a wide range of typical terrain characteristics to capture the diversity of topographic regions by prescribing five domain-averaged slope angles of 10°, 19°, 36°, 49°, and 59°. By prescribing \( \zeta \)'s ranging from 200 m to 1000 m in steps of 100 (for details, see Helbig and Löwe [2012, Table 4]). We employed fixed, squared model domain sizes (corresponding to a single grid cell size in a large-scale model) of \( L = 3 \) km and fixed (sub-)grid size \( \Delta x = \Delta y = 30 \) m resulting in square \( N \times N \) DEM's with \( N = 100 \).

Since the prescribed terrain parameters of our DEM's (\( \sigma \) and \( \xi \)) can only be exactly recovered by ensemble averaging over an unrealistic large number of realizations, we investigated which number of realizations would provide us with reasonably good statistics. For this we analyzed estimated covariance parameters with varying numbers of realizations. For a number of 200 realizations we found a maximum absolute error of only 7% for all \( \sigma, \xi \) combinations in the case of the steepest slope angle (59°). Using 200 realizations for each combination of \( \sigma, \xi \) we obtain a total of 9000 model topographies. All results given below for one combination \( \sigma, \xi \) actually represent an ensemble average over 200 realizations.

For each realization the parameters were estimated as follows. First, the valley-to-peak elevation difference \( \sigma \) was computed from the standard deviation of elevations \( \sigma_{\text{DEM}} \). Second, the mean-squared slope \( \mu \) was...
computed from equation (2) via finite differences. Subsequently, the typical width of topographic features \( \xi \) is computed from the relation \( \mu = \sqrt{2\xi} / \xi \) via

\[
\xi = \frac{\sqrt{2\sigma_{DEM}}}{\mu} = \frac{2\sigma_{DEM} \Delta x}{[(z_{j-1} - z_{j})^2 + (z_{j} - z_{j+1})^2]^{1/2}}.
\]

For completeness, we give the correct definition and the computation method of the sky view factor which was developed within the radiosity approach for the SW radiation balance [Helbig et al., 2009]. Even though an isotropic sky can be questioned especially under cloudy sky conditions, we here use the commonly applied assumption of isotropic sky radiation such that only one sky view factor is used for each subgrid cell location (for more details on anisotropic sky conditions see Dubayah and Rich [1995]). The sky view factor \( F_{\text{sky}} \), then is the proportion of the radiative flux received by an inclined surface from the visible part of the sky to that received from an unobstructed hemisphere. The subtle difference to the common “definition” of \( F_{\text{sky}} \) as the sky solid angle renders our method the only numerically exact scheme to compute \( F_{\text{sky}} \).

Domain-averaged sky view factors \( F_{\text{sky}}^\text{m} \) are computed for each topography from anisotropic terrain view factor sums by computing all view factors of each subgrid size. Note that view factors are purely geometric factors which were originally introduced by Siegel and Howell [1978] to describe the exchanges of radiant energy between surfaces in thermal engineering. For each view factor we solve the double area integral numerically with a uniform but adaptive areasubdivision of the two finite subgrid surface patches, i.e., the actual areas of each grid size (for computational details cf. [Helbig et al., 2009]). By neglecting boundary effects the numerical exact sky view factor \( F_{\text{sky}}^\text{m, ij} \), is obtained for each subgrid cell at location \( ij \) by seeing the sky as one large surface patch above the topography:

\[
F_{\text{sky}}^\text{m, ij} = 1 - \sum_{k,l} F_{\text{ijkl}}^\text{m}.
\]

The derivation of the sky view factor from anisotropic terrain view factor sums stems from the normalization property (energy conservation) of view factors in an enclosure. In the following we refer to domain-averaged values exclusively and therefore omit the overbar for notational convenience. Furthermore, numerically exact quantities are denoted by the superscript “m”, while we omit the superscript for parameterized quantities.

### 2.3. Sky View Factor Parameterization

Löwe and Helbig [2012] showed that the radiosity equation on GRF yields a theoretical framework enabling an analytical treatment of spatially averaged radiation transfer for isotropic topographies. In view of eliminating preprocessing subgrid topography to compute domain-averaged sky view factors \( F_{\text{sky}} \) a formally exact expression of the spatially averaged \( F_{\text{sky}} \) was derived. To carry out the averages over the random field

<table>
<thead>
<tr>
<th>Label</th>
<th>Geographic Location</th>
<th>( L ) (km)</th>
<th>( \Delta x ) (m)</th>
<th>( \zeta ) (deg)</th>
<th>( \mu )</th>
<th>( F_{\text{sky}}^\text{m} )</th>
<th>( \sigma ) (m)</th>
<th>( \xi ) (m)</th>
<th>( L / \xi )</th>
</tr>
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<td>3</td>
<td>30</td>
<td>31</td>
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<td>0.97</td>
<td>171</td>
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<td>0.97</td>
<td>169</td>
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<td>0.88</td>
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<td>33</td>
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<td>0.94</td>
<td>214</td>
<td>865</td>
<td>5.8</td>
</tr>
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</table>

*For U.S. models, see http://ned.usgs.gov. For models from the Alps, see http://www.swisstopo.ch. Given slope angles \( \zeta \), slope \( \mu \) (via equation (2)) and sky view factors \( F_{\text{sky}}^\text{m} \) (via equation (4)) are domain-averaged values. Given \( L \) values are mean values of the 2234 domain sizes.
Figure 1. (a) Relative errors between domain averages of numerically exact computed $F_{\text{sky}}^m$ (equation (4)) and parameterized $F_{\text{sky}}(\mu)$ (equation (5)) as a function of $L/\xi$ for the ensemble of model topographies (section 2). Symbols represent different domain-averaged slope angles (see section 2.2).

Figure 2. Sky view factors as a function of $L/\xi$ for the ensemble of model topographies (section 2). Symbols represent domain averages of numerically exact computed $F_{\text{sky}}^m$ (equation (4)) for each of the domain-averaged slope angles $\zeta$. Solid lines indicate parameterized values $F_{\text{sky}}(L/\xi, \mu)$ (equations (6) and (7)). The analytical approximations $F_{\text{sky}}(\mu)$ for $L/\xi \rightarrow \infty$ are shown by dotted lines for equation (5) and by dashed lines for the new approximation (equation (7)).

The joint probability density of horizon and slopes was reformulated in terms of a level-crossing problem of the underlying topography. The problem of computing radiation components was thus shifted to compute the level-crossing probability. Even though the level-crossing problem of Gaussian random surfaces cannot be solved exactly the resulting integrals can be evaluated by using an approximation which was previously derived for the scattering of sound waves at sea surfaces for GRF’s [Wagner, 1967]. For a wide range of domain-averaged slopes angles from about $7^\circ$ to $60^\circ$, an empirical fit formula for the sky view factor was derived:

$$F_{\text{sky}}(\mu) = \frac{1}{(1 + B \mu^b)^c},$$

with parameters $B = 4.4651$, $b = 2.00827$, and $c = 0.231192$ [Löwe and Helbig, 2012]. However, since the analysis was only carried out for $L/\xi \rightarrow \infty$ the derived sky view factor expression $F_{\text{sky}}(\mu)$ solely depends on the terrain parameter $\mu$ (equation (2)) for an isotropic slope distribution. Even though equation (5) forms a physically correct sky view factor parameterization it is strictly valid only for $L/\xi \rightarrow \infty$. However, with fixed, finite grid sizes, $L$, typically employed in large-scale models, and with varying widths of the underlying topographic features $\xi$ in each grid cell, a systematic correction for finite $L/\xi$ ratios is required for a reliable sky view factor parameterization. The deviations of realistic values $L/\xi$ from infinity for topographies from Switzerland and the U.S. become evident from the examples given in Table 1 in Helbig and Löwe [2012] and from the ones given here in Table 1.

3. Results

3.1. Sky View Factor Parameterization for Finite $L/\xi$ Ratios

We found large errors between sky view factors which are parameterized via $F_{\text{sky}}(\mu)$ (equation (5)) solely depending on slope $\mu$, and the numerically exact sky view factors $F_{\text{sky}}^m$ (equation (4)) for our model DEM’s (Figure 1). The errors decrease with increasing $L/\xi$ ratio and increase with increasing domain-averaged slope angle $\zeta$. Maximum errors for the largest mean slope angle $\zeta$ of $59^\circ$ are about 25%.

To derive a correction for finite $L/\xi$ ratios we statistically exploited that the domain averages of $F_{\text{sky}}^m$ computed from the large ensemble of model topographies, cover a variety of domain-averaged slopes and $L/\xi$ ratios. From a nonlinear regression analysis by robust M-estimators using iterated reweighted least squares (see R v2.14.1 statistical programming language [R Development Core Team, 2011] and its robustbase v0.8-0 package [Rousseeuw et al., 2011]) we obtain the following parameterization for spatially averaged sky view factors $F_{\text{sky}}(L/\xi, \mu)$ as a function of $L/\xi$ (from equation (3)) and $\mu$ (from equation (2))

$$F_{\text{sky}}(L/\xi, \mu) = 1 - (1 - F_{\text{sky}}(\mu)) e^{\frac{L/\xi}{\beta}}.$$  (6)
We investigated the influence of domain size.

3.2. Sensitivity to Domain Size

Computed from equation (5) (dotted lines) and from equation (6) (dashed lines) remain constant for all two scaling parameters, mean slope \( \mu \) and \( L/\xi \) ratio, is obvious from Figure 2 where the sky view factors computed from equation (5) (dotted lines) and from equation (6) (dashed lines) remain constant for all \( L/\xi \). Sky view factors computed from equation (6) provide a better fit for large \( L/\xi \) ratios than sky view factors from equation (5).

3.2. Sensitivity to Domain Size

We investigated the influence of domain size \( L \) on the sky view factor parameterization by means of a second ensemble of model topographies. For this we generated an ensemble of GRF as model topographies with a prescribed domain-averaged slope angle of 36°, a fixed, squared domain size \( L = 6 \) km and a prescribed \( \xi \) of 900 m and \( \sigma \) of 382 m. We subdivided these topographies in subdomains with domain size \( L = 3 \) km and \( \Delta x = \Delta y = 30 \) m was used for all domains. With 200 realizations of each \( \sigma, \xi \) combination this results in a total of 600 topographies. Table 2 shows that subdividing the domain size \( L \) alters the actual characteristic width \( \xi \) (derived via equation (3)) and height \( \sigma \) (standard deviation of elevations) of topographic features, as well as the mean sky view factor. Subdividing the domain size did, however, barely change the domain-averaged slope. Differences between parameterized and numerically exact computed \( F_{\text{sky}} \) were negligible for all \( L/\xi \) ratios with maximum differences of 0.0013 for a \( L/\xi \) ratio of 2.6 with \( L = 1.8 \) km.

### Table 2. Sky View Factors for Model Topographies With Three Domain Sizes \( L \) Subdivided From Topographies With \( L = 6 \) km, a Prescribed \( \xi = 900 \) m and \( \sigma = 382 \) m (section 3.2)*

<table>
<thead>
<tr>
<th>( L ) (km)</th>
<th>( F_{\text{sky}}^n )</th>
<th>( F_{\text{sky}}(L/\xi, \mu) )</th>
<th>( \xi ) (m)</th>
<th>( \sigma ) (m)</th>
<th>( \xi ) (deg)</th>
<th>( L/\xi )</th>
</tr>
</thead>
<tbody>
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<td>0.868</td>
<td>0.869</td>
<td>877</td>
<td>372</td>
<td>64.30</td>
<td>6.84</td>
</tr>
<tr>
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<td>0.902</td>
<td>796</td>
<td>340</td>
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<td>0.938</td>
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<td>295</td>
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</tbody>
</table>

*For domain-averaged numerically exact computed sky view factors \( F_{\text{sky}}^n \), see equation (4). For parameterized sky view factors \( F_{\text{sky}}(L/\xi, \mu) \), see equations (6) and (7). Gaussian covariance parameters \( \sigma \) (\( \sigma_{\text{DEM}} \)) and \( \xi \) were obtained from equation (3). For domain-averaged slope angles \( \mu \), see equation (2). The choice of the functional form was motivated as follows: For \( L/\xi \to \infty \) we want to recover the form suggested by the analytical treatment presented in equation (5) [Löwe and Helbig, 2012], potentially allowing for different parameters. For finite \( L/\xi \) we expect a correction which include Gaussian factors \( e^{-\varpi^2/\xi^2} \) due to dimensional requirements and the Gaussian nature of the problem. This is, however, not limiting the application to topographies showing Gaussian slope statistics which will be shown later. Compared to the previous fit obtained from the analytical treatment (equation (5)) we used the same functional form for \( F_{\text{sky}}(\mu) \) but estimated slightly different parameters in equation (6), namely

\[
F_{\text{sky}}(\mu) = \frac{1}{(1 + B' \mu^c)^c}.
\]

with parameters \( B' = 3.354688 \), \( b' = 1.998767 \), and \( c' = 0.20286 \). Our new subgrid sky view factor parameterization as in equations (6) and (7) predicts the numerically exact results for our large ensemble of DEM’s very well (Figure 2). We obtain an overall root-mean-square error (RMSE) of 0.001092. The relevance of the two scaling parameters, mean slope \( \mu \) and the \( L/\xi \) ratio, is obvious from Figure 2 where the sky view factors computed from equation (5) (dotted lines) and from equation (6) (dashed lines) remain constant for all \( L/\xi \). Sky view factors computed from equation (6) provide a better fit for large \( L/\xi \) ratios than sky view factors from equation (5).

### Figure 3. Sky view factors as a function of the standard deviation for isotropic slopes \( \mu \) (equation (2)) for all model topographies (section 2). Black symbols represent domain averages of numerically computed \( F_{\text{sky}}^n \) (equation (4)) for the model topographies with five domain-averaged slope angles \( \xi \) (see section 2.2) as well as for the real topographies (Table 1). Red plus signs denote parameterized sky view factor \( F_{\text{sky}}(L/\xi, \mu) \) with equations (6) and (7). The solid line indicates the analytical approximation \( F_{\text{sky}}(\mu) \) for \( L/\xi \to \infty \) (equation (7)).
Figure 4. Relative errors between domain averages of numerically exact computed sky view factors $F_{\text{sky}}^m$ (equation (4)) and parameterized sky view factors $F_{\text{sky}}(L/\xi, \mu)$ with equations (6) and (7) as a function of $L/\xi$ for all model topographies (crosses) (section 2) and real topographies (squares) (Table 1). The mean $F_{\text{sky}}$ error for all real topographies was 2.6%. Second, we validated the parameterization systematically by computing numerically exact values for Switzerland. We computed $F_{\text{sky}}^m$ for entire Switzerland (approximately $385 \times 225$ km$^2$) from a DEM (http://www.swisstopo.ch/en) with a horizontal resolution $\Delta x = \Delta y = 25$ m. For this we employed a distance restriction in the anisotropic terrain view factor sums (equation (4)) to view factors in distances less than 5 km for each grid cell which still amounts to a significant computation time of a few months, even on several 2.8 GHz AMD Opteron (12 CPU cores) compute nodes used in parallel analysis for spatially independent parts of Switzerland. The averaged sky view factor of entire Switzerland is 0.942 with individual values ranging from a minimum of 0.2 to the maximum of one. For the validation, we subsequently averaged the subgrid horizontal resolution of $\Delta x = 25$ m in squared domain sizes of 1 km (with $40 \times 40$ subgrid cells in each), 2.5 km (with $100 \times 100$ subgrid cells in each) and 5 km (with $200 \times 200$ subgrid cells in each) and compared the domain-averages obtained numerically to parameterized $F_{\text{sky}}$ for the corresponding domain size (Figure 5) via scatter plots which show the performance of the parameterization for the topography of Switzerland. The RMSE for the new subgrid sky view factor parameterization depend on $L$ and increase with decreasing domain size (RMSE$^1_{5 \text{km}} = 0.012$, RMSE$^2_{2.5 \text{km}} = 0.027$, and RMSE$^1_{1 \text{km}} = 0.047$). Errors also decrease with increasing $L/\xi$ ratio (Figures 5 and 6) as well as with decreasing domain-averaged slope $\mu$ (Figure 6). Overall, the parameterization works very well for real terrain of Switzerland subdivided in domain sizes of

Figure 5. Domain averages of numerically exact computed sky view factors $F_{\text{sky}}^m$ (equation (4)) as a function of parameterized sky view factors $F_{\text{sky}}(L/\xi, \mu)$ with equations (6) and (7) for Switzerland. $F_{\text{sky}}^m$ is computed at $\Delta x = 25$ m and subsequently averaged for three domain sizes $L = 5$ km, 2.5 km and 1 km. Colors represent different $L/\xi$ ratios.
thetwotopographiclengthscales,thetypicalvalley-to-peaked elevation difference $\sigma$, also simply a consequence of dimensional analysis: The GRF on the infinite domain is fully characterized by $F$ less quantity like $\sigma$.

We validated the parameterization in two steps. First, we compared parameterized sky view factors to model topographies. We presented a subgrid parameterization for sky view factors over complex topography based on investigating numerically exact computed sky view factors on isotropic Gaussian Random Fields (GRF) as model topographies. The parameterization only involves computational efficient terrain parameters which are the mean-squared slope and the correlation length of subgrid topographic features. Any tedious preprocessing of the subgrid topography to compute nonlocal horizon angles became redundant. Though, such a preprocessing has to be carried out only once in applications, this can lead to a significant numerical effort (cf. section 3 for the example of Switzerland). The parameterization can be applied in radiation schemes of large-scale meteorological, land surface, and climate models or for the correction of satellite imagery from coarse scale swath angles over mountainous terrain.

To develop the parameterization we generalized a previously derived analytical sky view factor approximation $F_{\text{sky}}(\mu)$ (Löwe and Helbig, 2012), solely dependent on mean-squared slope $\mu$ (equation (2)), by introducing the $L/\xi$ ratio as a second scaling parameter. This was necessary as $F_{\text{sky}}(\mu)$ in Löwe and Helbig [2012] was strictly valid only for $L/\xi \to \infty$, i.e., $F_{\text{sky}}(\mu)$ does not change with variations in typical widths of the underlying topographic features $\xi$ or with the domain size $L$. However, typical values of $L/\xi$ are far from being infinite (cf. Table 1 for typical real topography characteristics) and large errors result without including the $L/\xi$ ratio in the parameterization (cf. Figures 1 and 2). Due to the linear dependency of $F_{\text{sky}}$ in the subgrid parameterizations describing limited sky view in the shortwave (SW) and longwave (LW) radiation (e.g., Helbig and Löwe, 2012) the errors in the sky view factor parameterization strongly influence the overall accuracy of a subgrid radiation parameterization scheme. Note that the necessity of including $L/\xi$ for GRF is also simply a consequence of dimensional analysis: The GRF on the infinite domain is fully characterized by the two topographic length scales, the typical valley-to-peak elevation difference $\sigma$ and $\xi$. Any dimensionless quantity like $F_{\text{sky}}$ must therefore be expressible as a function solely depending on $\sigma/\xi$. A finite domain, however, introduces a third length scale $L$ and dimensional analysis dictates that $F_{\text{sky}}$ must be a function of two arguments, namely both length scale ratios $\sigma/\xi$ and $L/\xi$. Furthermore, a significant variance of domain averages of multiple terrain reflections with the $L/\xi$ ratio of topographies was previously found [Helbig et al., 2009] indicating the need of $L/\xi$ as a second scaling parameter. For the derivation of the new subgrid parameterization valid for finite $L/\xi$ ratios we statistically exploited domain-averaged sky view factors computed from anisotropic terrain view factor sums (equation (4)) for a large, diverse ensemble of 9000 GRF as model topographies.

We validated the parameterization in two steps. First, we compared parameterized sky view factors to domain-averaged numerically exact computed sky view factors (via equation (4)) for a set of 14 real topographies from Switzerland and the U.S. We found maximum relative errors of only 7.5% with a mean relative error of 2.6% (Figures 3 and 4). Note that the range of the number of topographic features per domain size of these real topographies ($L/\xi$ ratios of 4.1 to 7.9, cf. Table 1) confirms our choice of the range covered by our large ensemble of model topographies for the derivation of the sky view factor parameterization ($L/\xi$ ratios of 3 to 15). Second, we validated parameterized sky view factors systematically for a large region: the entire Switzerland. Sky view factors were computed at a horizontal resolution of 25 m. By averaging over various squared domain sizes, we obtained in total 2234 usable computed values for a domain size of 5 km, 8862 values for a domain size of 2.5 km, and 57,154 for a domain size of 1 km (Figure 5). Overall, the parameterization agreed well with numerical values, especially for the two largest domain sizes of 2.5 km and 5 km.

![Figure 6. Relative error between domain averages of numerically exact computed sky view factors $F^{m}_{\text{sky}}$ (equation (4)) and parameterized sky view factors $F_{\text{sky}}(L/\xi, \mu)$ with equations (6) and (7) for Switzerland with a domain size $L = 5$ km as a function of domain-averaged slopes $\mu$ (equation (2)). Colors represent different $L/\xi$ ratios.](image-url)
Thus, with increasing grid sizes (or domain size here), we expect an even lower scatter. The overall small RMSE for each domain size outline the good performance of our parameterization (RMSE5 km = 0.012, RMSE2.5 km = 0.027, and RMSE1 km = 0.047).

We briefly discuss possible error sources for the remaining differences for Switzerland: A very small RMSE of 0.0011 was obtained for the new parameterization on GRF as topographies (Figure 2). Even though we previously found that Gaussian statistics are well suited to approximate the slope statistics of real topography [Helbig and Löwe, 2012] there is some evidence that the scaling properties of the Swiss topography cannot be fully described by a Gaussian covariance. A previous analysis e.g., revealed power law characteristics [Dietler and Zhang, 1992]. In principle, this power-law behavior could be met within GRF by adopting a different covariance for the model topographies. However, the main features of topography are already well captured by the simpler covariance (equation (1)) as demonstrated by the accuracy of the results. In addition, the terrain characteristics from Switzerland are reasonably well described by our ensemble of GRF model topographies [see Helbig and Löwe, 2012, Table 4]. Another issue worth discussing are the slightly different parameters in the sky view factor expression $F_{\mu}(\omega)$ (equation (7)) in contrast to the parameters obtained from the analytical approach as in equation (5). The derivation from Löwe and Helbig [2012] is based on the shadowing probability from Wagner [1967] which employs the statistical independence of slopes and heights of the GRF at some point. Slope and heights are, however, correlated for GRF with covariance as in equation (1). This is naturally accounted for in the present approach, and this difference will likely cause the slight deviations in the fit parameters. Lastly, a minor error contribution might stem from a distance restriction in the numerically exact computation of sky view factors for Switzerland. As a first approach, we only employed those terrain view factors in the anisotropic terrain view factor sums (equation (4)) that were in a distance smaller than 5 km from each 25 × 25 m² grid cell which was done to reduce the vast computation time.

Overall, sky view factor errors clearly decrease with increasing $L/\xi$ ratio for entire Switzerland (Figure 5) outlining the $L/\xi$ ratio as a relevant second scaling parameter. For $L/\xi$ ratios larger than around 10 the errors are significantly reduced. Note that a large $L/\xi$ ratio indicates that more terrain is included in the model domain ($L \times L$) and thus describes the subgrid topographic features more accurately. This also coincides well with a previously found result that the $L/\xi$ ratio must be sufficiently large to reduce significant effects of the finite domain size [Helbig et al., 2009]. Interestingly, for the topography from Switzerland, averaged over squared domain sizes $L$ of 5 km, we found that the typical height of topographic features $\sigma$ is consistently smaller than the typical width of topographic features $\xi$. This means that we could not find very steep domain-averaged slopes (> 49°) which would imply a $\sigma \approx \xi$ (cf. Figure 6).

Subdividing domain sizes $L$, investigated on an additional ensemble of 600 model topographies generated with a domain size of 6 km, demonstrated a large impact on sky view factors (Table 2). We considered the influence of variable domain sizes since the derivation of the sky view factor parameterization was based on accurately computed sky view factors on model topographies employing a fixed domain size of 3 km. We found that smaller domain sizes still result in very well parameterized sky view factors with maximum differences of 0.0013 for a $L/\xi$ ratio of 2.6 with $L = 1.8$ km (cf. Table 2). Note that subdividing the same topographies lead to different actual characteristic widths (derived via equation (3)) and heights (standard deviation of elevations) of topographic features but roughly the same domain-averaged slopes demonstrating the importance of the $L/\xi$ ratio as a second scaling parameter.

Previously, Löwe and Helbig [2012] developed a quasi-analytical expression for the domain-averaged sky view factor to solely describe subgrid topographic impacts on both the SW and LW surface radiation balance, i.e., limited sky view and terrain reflections/emissions. Here we generalized this analytical approximation for applications in fixed, finite large-scale grid sizes $L$ with varying subgrid topographic correlation lengths $\xi$. Given the increasing interest of SW surface radiation maps for e.g., climate monitoring, planning of solar energy applications, model verification purposes, etc., this new subgrid parameterization for sky view factors in complex terrain is, however, not only of interest for large-scale meteorological, land surface, and climate models but also for the terrain correction in satellite retrieved SW surface radiation maps. Geostationary satellite sensors provide a continuous spatial and temporal coverage to derive SW radiation maps [Stöckli, 2013]. However, the derivation of SW radiation maps from satellite signals requires complex algorithms extracting clouds and the influence of topography on the signal [Stöckli, 2013]. Until now, topographic influences are only accounted for on resolved topography. The sky view factor parameterization...
presented here, could, together with the subgrid parameterization scheme for SW surface radiation published previously by Helbig and Löwe [2012] and Löwe and Helbig [2012], help to fill this gap by providing the reference cloudless sky radiation accounting for all subgrid topographic impacts.

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