Stochastic modeling of the cover effect and bedrock erosion

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[1] Several important fluvial bedrock erosion processes are driven by the impact of bed load particles. Bed load transport rates fluctuate strongly in both nature and experiment, and stochastic models of the transport processes have been put forward to describe this behavior. In this paper I adapt a model based on a Markov chain formulation to derive probability distributions for bed load transport rate over a rock bed only partly covered by sediment. I propose a way to calculate the probability distribution of bed cover for given sediment supply using a combinatoric model and combine the two curves to calculate probability distributions for bed cover and erosion rate at constant hydraulics. In the proposed model, mean bed cover is an exponentially declining function of the number of particles in the control volume. The model describes recently published experimental data well, but at the moment it is not possible to finally discriminate between the exponential and the previously proposed linear model formulation. Distributions of erosion rate are fairly broad functions slightly skewed toward high erosion rates.


1. Introduction

[2] Bedrock rivers play a key role in landscape evolution. They erode bedrock in the channel, provide the base level for hillslope response, and evacuate the sediment produced within the catchment [Whipple, 2004]. The sediment transported by a river has two conflicting effects on bedrock erosion [Sklar and Dietrich, 1998, 2004]: An increasing amount of moving sediment increases the number of impacts on the bed and therefore the erosion rate (the tools effect). On the other hand, the sediment within the stream covers a fraction of the rock bed and thereby protects it, decreasing the erosion rate (the cover effect). Although it has long been suggested that particle impacts drive the dominant types of bedrock erosion [Gilbert, 1877], only recently have sediment flux effects been fully incorporated into fluvial erosion models [Sklar and Dietrich, 1998, 2004; Whipple and Tucker, 2002]. Sklar and Dietrich [2004] have developed a mechanistic model of bedrock erosion by saltating bed load, in which the erosion rate $E$ is the product of three terms, describing (1) the volume of material removed at each particle impact $V_i$; (2) the rate of impact of particles per unit area $I_i$; and (3) the fraction of bedrock exposed to the flow $R_a$:

$$E = V_i I_i R_a.$$

(1)

It has since been demonstrated that sediment transport, bedrock erosion, and channel morphology are tightly coupled both in experiments [Finnegan et al., 2007; Johnson and Whipple, 2007; Shepherd, 1972; Sklar and Dietrich, 2001; Wohl and Ikeda, 1997] and in nature [Cowie et al., 2008; Turowski et al., 2008a, 2008b]. Although their results cannot directly be linked to erosion rates, Turowski and Rickenmann [2009] demonstrated that both tools and cover effect are active in the interaction of sediment particles and the channel bed in a mountain stream in Austria.

[3] The cover effect is often described by a linear model [Sklar and Dietrich, 1998, 2004], where

$$R_a = \begin{cases} 1 - \frac{Q_t}{Q_s} & Q_t < Q_s, \\ 0 & Q_t \geq Q_s. \end{cases}$$

(2)

Here $Q_t$ is the bed load flux in the channel and $Q_s$ is the bed load transport capacity. Chatanantavet and Parker [2008] studied the development of bed cover as a function of flow rate, sediment supply, and bed slope in a laboratory flume, and concluded that equation (2) provides a good approximation to their observations. Because channel bed roughness can differ between alluviated stretches and bare bedrock, in their experiments equilibrium states may be dependent on the distribution of sediment at the start of a run. In addition, the bed topography plays an important role. Turowski et al. [2007] put forward an exponential decline as the first theoretically motivated equation for bed cover, which takes the form

$$R_a = \begin{cases} \exp\left(-\frac{\varphi Q_t}{Q_s}\right) & Q_t < Q_s, \\ 0 & Q_t \geq Q_s. \end{cases}$$

(3)

In the model, the cover factor $\varphi$ is a dimensionless parameter with yet unknown dependence on bed topography and roughness and determines whether sediment accumulates in certain regions ($\varphi < 1$) or spreads out ($\varphi > 1$). For a...
flat bed with evenly distributed deposition probabilities, it is expected to be equal to one. Equation (3) has not been systematically tested, but it provides a better fit to the erosion data reported by Sklar and Dietrich [2001] than equation (2) (Figure 1). Turowski et al. [2007] also introduced the concept of dynamic cover, in contrast to static cover due to particles resting on the bed without motion. Dynamic cover arises as grain-grain interactions become more frequent with increasing concentration of moving sediment, reducing the impact rate on the bed. In their model, only dynamic cover occurs for \( Q_s < Q_n \), since by definition all sediment is mobile.

[4] It is clear that a thorough understanding of sediment transport processes is essential to modeling bedrock erosion. Bed load transport rates observed in the laboratory [e.g., Böhm et al., 2004; Frey et al., 2003; Hassan and Church, 2001; Kühne and Southard, 1988] or in nature [e.g., Gomez and Church, 1989; Hegg and Rickenmann, 1998; Méthivier et al., 2004] often show strong fluctuations, even at constant hydraulic conditions. This has been attributed to various effects, for example, to the passing of individual bed forms or bed material waves [Gomez et al., 1989; Lisle et al., 2001], the temporal variations in bed armouring and longitudinal sorting [Gomez, 1983; Iseya and Ikeda, 1987], the temporal and spatial variation of the distribution of grain sizes and of grain arrangement [Chen and Stone, 2008; Cudden and Hoey, 2003; Kirchner et al., 1990], and temporally varying sediment supply conditions [Benda and Dunne, 1997; Warburton, 1992]. Some of the earliest theoretical approaches describing bed load transport were formulated as stochastic theories [Einstein, 1937, 1950; Kalinske, 1947]. These ideas have since been developed further [e.g., Ancey et al., 2006, 2008; Hegg and Rickenmann, 1998; Kleinhans and van Rijn, 2002; Paintal, 1970]. Various deterministic models have also been proposed [e.g., Ashida and Michiue, 1972; Bagnold, 1956, 1977; Engelund and Fredsøe, 1976], and one of the most frequently cited of these is Bagnold’s [1956, 1977] so-called mean-field theory, which is based on a formulation of the momentum exchange between the particles and the fluid. However, after Niño and García [1994] already reported problems with Bagnold’s [1956, 1977] theory, Seminara et al. [2002] showed that an approach based on this model leads to internal contradictions when generalized to arbitrarily sloping beds and can therefore not be generally true. On the contrary, a stochastic model based on the one developed by Einstein [1950] has been successfully applied to arbitrarily sloping beds [Parker et al., 2003]. Thus the evidence increases that a stochastic formulation is necessary to completely describe bed load transport processes.

[5] In this paper I adapt a stochastic transport model proposed by Ancey et al. [2008] to describe sediment transport over a rock bed. The model can be used to derive probability distributions for sediment flux and storage, bed cover, and bedrock erosion rate. Since data sets on bedrock erosion with high temporal resolution do not currently exist, I focus on the transport process and on the cover effect in particular. The model predictions for the mean bed cover are compared to published experimental results.

2. Model Setup and Derivation

[6] Since many factors influence erosion rates, a stochastic model of bedrock erosion results from the combination of the distribution functions of the various control parameters. For example, at constant hydraulic conditions one would expect both the number of particles and their velocities to fluctuate to some extent. This affects both the impact rate and the bed cover. Furthermore, not every impact will cause the same amount of erosion, for example, due to different particle sizes and impact energies or due to heterogeneities in bedrock erodibility or crack density. Thus
2.1. Number of Particles

2.1.1. Model Setup

The number of particles in motion as bed load. Material is supplied at the rate \( \lambda_0 \) and is evacuated at the rate \( \nu \). It can be entrained by hydraulic processes with the rate \( m_\lambda \) and by collective entrainment through the impact of other particles at the rate \( m_{\nu} \). Mobile particles are deposited at the rate \( \sigma \).

The final probability density function (pdf) describing the erosion rate is given by

\[
\text{pdf}(E) = \text{pdf}(V) \ast \text{pdf}(L) \ast \text{pdf}(R),
\]

and the mean erosion rate is given by equation (1). Here the asterisk denotes the convolution operation, which is defined as

\[
f(x) \ast g(x) = \int_{-\infty}^{\infty} f(y)g(x-y)dy.
\]

Here \( f(x) \) and \( g(x) \) are functions of \( x \), and \( y \) is a dummy variable. The pdf’s on the right-hand side in equation (4) may again be convolutions of pdf’s of more fundamental variables. For example, the first term incorporates variations in substrate and grain properties, and impact energy, the second term incorporates the variability of saltation hop length (and thus the number of impacts of a given particle) and sediment availability, and the third term incorporates fluctuations in bed cover, which is dependent on the number of particles in a control volume and their distribution [Sklar and Dietrich, 2004]. Particle motion and entrainment are controlled by turbulent forces, and thus local fluid velocity, which can also fluctuate strongly [e.g., Schmeckle et al., 2007; Sumer et al., 2003]. In the following I shall deal in turn with the pdf’s of several of the control parameters.

2.1. Number of Particles

2.1.1. Model Setup

In stochastic transport models, the sediment transport arises from the imbalance between entrainment and deposition. The transport rate can be defined as the number of bed load particles crossing a surface within a given period of time. Alternatively, it can be useful to define the flux of bed load particles crossing a surface within a given period. The transport rate can be defined as the number of particles at the rate \( m \lambda \) and by collective entrainment through the impact of other particles at the rate \( m_{\nu} \). Mobile particles are deposited at the rate \( \sigma \).

\[
P(m; n \rightarrow n+1; \Delta t) = \lambda_0 \Delta t.
\]

\[
P(m; n \rightarrow n-1; \Delta t) = \nu \Delta t.
\]

\[
P(m \rightarrow m-1; n \rightarrow n+1; \Delta t) = m_\lambda \Delta t + m_{\nu} \Delta t.
\]

\[
P(m \rightarrow m+1; n \rightarrow n-1; \Delta t) = \sigma \Delta t.
\]

Here \( P(m \rightarrow x, n \rightarrow y; \Delta t) \) denotes the probability of a transition of \( m \) going to \( x \) and \( n \) going to \( y \) in the time interval \( \Delta t \).

Since the current state gives a complete description of the system, the Markov property (the state at \( t + \Delta t \) depends only on the state at time \( t \) and not, for example, the state at time \( t - \Delta t \)) is satisfied and the Markov chain formalism can be used to set up the model. Following the normal methods of setting up the equations of a forward Markov chain [e.g., Bolch et al., 2006], the probability of observing a state with \( m \) stationary and \( n \) moving particles at time \( t + \Delta t \) can be derived. Let \( (m, n; t) \) denote the state with \( m \) stationary particles and \( n \) moving particles at time \( t \), and let \( P(m, n; t) \) denote the probability to observe this state. If one assumes that \( \Delta t \) is short enough such that only a single transition can occur (i.e., the probability that several transitions occur within \( \Delta t \) is negligible), the state \( (m, n; t + \Delta t) \) can be accessed from the neighboring states \( (m-1, n; t) \) (upstream supply), \( (m+1, n+1; t) \) (downstream evacuation), \( (m-1, n-1; t) \) (entrainment), and \( (m-1, n+1; t) \) (deposition). The probability to observe state \( (m, n; t + \Delta t) \) can then be written as the sum of the probabilities for the neighboring states multiplied by the respective transition probabilities given in equations (6)–(9). In addition, one has to take into account the possibility that no transition
occurs within $\Delta t$ and both $m$ and $n$ remain unchanged. The probability $P(m, n; t + \Delta t)$ is then

$$P(m, n; t + \Delta t) = P(m, n - 1; t)\lambda_0 \Delta t + P(m + 1, n - 1; t)$$

$$+ (m + 1)\lambda_1 + (m + 1)(n - 1)\mu_1) \Delta t + P(m - 1, n + 1; t)(n + 1)\sigma \Delta t$$

Rearranging the terms and letting $\Delta t \to 0$, the master equation describing the time evolution of the system can be written as

$$\partial P(m, n; t)/\partial t = \lambda_0 P(m, n - 1; t) + ((m + 1)\lambda_1$$

$$+ (m + 1)(n - 1)\mu_1)P(m + 1, n - 1; t) + (n + 1)\sigma P(m - 1, n + 1; t) + (n + 1)\nu P(m, n + 1; t)$$

$$- (\lambda_0 + m\lambda_1 + n\nu + n\sigma + mn\mu)P(m, n; t).$$  \hspace{1cm} (10)

### 2.1.2. Boundary Conditions

For a numerical solution of equation (10) the boundary conditions need to be specified. A solution exists when both $m$ and $n$ are nonnegative integers. Boundary conditions thus need to be specified for all points $m = 0$, and $n = 0$, and in the limits $m \to \infty$ and $n \to \infty$. Furthermore, the initial condition at $t = 0$ needs to be given.

In the difference formulation of equation (10) the zero-state conditions read

$$P(m, 0; t + \Delta t) = P(m - 1, 1; t)\sigma \Delta t + P(m, 1; t)\nu \Delta t$$

$$+ P(m, 0; t)[1 - \Delta t(\lambda_0 + m\lambda_1)]$$  \hspace{1cm} (12)

$$P(0, n; t + \Delta t) = P(0, n - 1; t)\lambda_0 \Delta t + P(1, n - 1; t)$$

$$\cdot [\lambda_1 + (n - 1)\mu_1) \Delta t + P(0, n + 1; t)(n + 1)\nu \Delta t$$

$$+ P(0, n; t)[1 - \Delta t(\lambda_0 + n\nu + n\sigma)]$$  \hspace{1cm} (13)

$$P(0, 0; t + \Delta t) = P(0, 1; t)\nu \Delta t + P(0, 0; t)[1 - \lambda_0 \Delta t].$$  \hspace{1cm} (14)

At time $t = 0$ there are $M_0$ and $N_0$ particles in the control volume. Therefore

$$P(m, n; 0) = \begin{cases} 1 & m = M_0, \ n = N_0 \\ 0 & \text{otherwise} \end{cases}.$$

In the limit when $m$ and $n$ go to infinity, the probability of occurrence needs to be zero regardless of the value of the other parameter, and thus the following boundary conditions apply:

$$\lim_{m \to \infty} P(m, n; t) \to 0,$$  \hspace{1cm} (16)

$$\lim_{n \to \infty} P(m, n; t) \to 0.$$  \hspace{1cm} (17)

I used equation (10) together with the boundary conditions (12)–(17) to calculate probabilities for each combination of $m$ and $n$, giving a joined probability distribution for the two parameters. Practically, instead of infinite $m$ and $n$, for conditions (16) and (17) maximal values of $m$ and $n$ were chosen, which were much larger than the expected values of $m$ and $n$ at steady state, denoted by $m_s$ and $n_s$, respectively. At these values $P(m, n; t)$ is negligible and was set to zero for all $t$.

### 2.1.3. Results

[13] Ancey et al. [2006, 2008] derived a similar model as the one described above for bed load transport over an alluviated bed (in effect, they assumed that $m$ approaches infinity; see also Appendix A). In parallel, they studied the stochastic properties of sediment transport in a flume just wide enough for a single particle. This approach allowed them to track individual particles and their entrainment and deposition, and thus to constrain the rate constants for their system. The experiments by Ancey et al. [2006, 2008] are strongly simplified versions of reality; therefore their rate constants cannot be directly applied to real rivers. However, as no other constraints are available, I will use values similar to the ones derived for experiment (a) by Ancey et al. [2008] as order of magnitude estimates for the values of the rate constants (case S for small, Table 1). Since these values produce a distribution with the average value of $m$ close to zero (and thus a partly cutoff distribution function; Figure 3), I give a second example using a higher value for the deposition parameter $\sigma$ (case L for large). This results in larger $m$ values on average, and the complete distribution function can be seen (Figure 3). For case L, the base of the distribution is elongated, with the short axis aligned in the direction of both increasing $m$ and $n$ (although not exactly at 45°). For case S, the distribution is cut off, since the mean value of $m$ is close to zero. This results into a strong skew toward small values of $m$. The distributions are fairly narrow in both cases, and the probability of occurrence is close to zero for most of the parameter space. For example, for case S, states with $m > 5$ can safely be neglected, compared with a mean $m \approx 1.13$ (Figure 3).

[14] A simple flux balance gives equations for the expected values of $m$ and $n$ at steady state, $\bar{m}_s$ and $\bar{n}_s$, respectively. Since influx must balance outflux, $\bar{n}_s$ is given by

$$\bar{n}_s = \frac{\lambda_0}{\nu},$$  \hspace{1cm} (18)

while a balance for the expected number of stationary particles at steady state $\bar{m}_s$ is given by the equation

$$\bar{m}_s = \frac{\bar{n}_s \sigma}{\lambda_1 + \bar{n}_s \mu} = \frac{\sigma \lambda_0}{\nu \lambda_1 + \mu \lambda_0}.$$  \hspace{1cm} (19)
However, while equation (18) exactly predicts the model results (Figure 4a), in general the calculated $\bar{m}_s$ is slightly larger than predicted by equation (19) (Figure 4b). The simple balance equation (19) implicitly assumes that the local probability exchange for each state $m, n$ at steady state is symmetric. Mathematically, this means that for every state the deposition rate equals the entrainment rate and the supply rate equals the evacuation rate. Since the collective entrainment term $m n \mu$ is dependent on both $m$ and $n$, the symmetry is broken. The probability exchanges are not locally balanced (i.e., for each state separately), but only over the whole parameter space under consideration. An observable result of this is that the axes of symmetry of the distribution functions are not parallel to lines with constant $m$ and constant $n$, but are at an angle to these (compare Figure 3). For the case $\mu = 0$, $\bar{m}_s = \lambda_0 \sigma / \lambda_1 \mu$, as predicted by equations (18) and (19). It was not possible to find a simple formula to predict $\bar{m}_s$ from the values of the rate constants. The variation of $\bar{m}_s$ with changing rate parameters is shown in Figure 5. For the calculations, each of the rate parameters was varied between $1 \text{ s}^{-1}$ and $100 \text{ s}^{-1}$. To show the variation that can be obtained in the model response, I used five sets of values for the other parameters: In the first set, each of the parameters was set to $10 \text{ s}^{-1}$; in the other sets, one of the four constant parameters was set to $50 \text{ s}^{-1}$ while the others remained at $10 \text{ s}^{-1}$. The values for $\bar{m}_s$ were obtained by running each simulation to a steady state and then calculating the average $m$ for the predicted distribution function. The behavior of $\bar{m}_s$ with changing parameter values depends on the other parameters and can vary considerably. For example, $\bar{m}_s$ increases monotonically with

![Figure 3](image_url). Steady state probability distributions for the example values of the rate constants (Table 1) for (top) case L ($\bar{m}_s = 24.8, \bar{n}_s = 13.3$) and (bottom) case S ($\bar{m}_s = 1.13, \bar{n}_s = 13.3$). Note the different scales both on the $m$ and the probability axis.

![Figure 4](image_url). Mean $m$ and $n$ at steady state for $\sim 1800$ model runs with different parameter values. The value calculated from equations (18) and (19) is shown on the $y$ axis, and the value calculated from the modeled distribution functions is shown on the $x$ axis. While equation (18) predicts the mean number of moving particles $\bar{n}_s$ accurately (Figure 4a), equation (19) overestimates the mean number of stationary particles $\bar{m}_s$ in general (Figure 4b). This is due to an asymmetry in the probability exchange.
Figure 5. Variation of the mean number of stationary particles at steady state $\bar{m}$ with the rate parameters. For all calculations the control parameters were set to $10 \text{ s}^{-1}$, except when stated otherwise in the legend. For comparison, the prediction using equation (19) is also given (dotted line).
Equations for bed cover have previously been proposed as deterministic models for steady state conditions when the bed configuration has already adjusted to supply conditions [Sklar and Dietrich, 1998; Turowski et al., 2007]. Turowski et al. [2007] used a probabilistic argument to derive an exponential equation for bed cover. Here I shall use a similar line of reasoning but model the stochastic variation explicitly. As such, the model proposed below provides a stochastic interpretation for the exponential model derived by Turowski et al. [2007].

In general, there are two sources of variation for bed cover: (1) At steady hydraulic conditions, the extent of cover follows the stochastic variation of the variation of the number of particles within the control volume. This fluctuation was treated in section 2.1. (2) At any instant, the bed in the control volume is populated by a total number of \( k = m + n \) particles, which may take varying positions on the bed within the control volume. The fraction of bedrock exposed for given number of particles \( k \) can be modeled as follows. The particles within the control volume are randomly distributed into \( N \) cells, each of which is completely covered when a single particle resides there. The fraction of exposed bed is then \( x/N \), where \( x \) is the number of empty cells. Thus the problem is equivalent to distributing \( k \) balls into \( N \) boxes and asking for the probability \( P_k^N(x) \) that exactly \( x \) boxes remain empty. Assuming that the number of balls that fit into a box is unlimited, the probability is given by the recursive relation

\[
P_k^N(x) = \frac{N - r}{N} P_{k-1}^N(x = r) + \frac{r + 1}{N} P_{k-1}^N(x = r + 1).
\]

The probabilities for \( k = 0 \) are given by

\[
P_0^N(x) = \begin{cases} 0 & 0 \leq r < N, \\ 1 & r = N. \end{cases}
\]

Equations (20) and (21) have been used to calculate the probability distribution for bed cover at given \( k \). To obtain the probability distribution of the fraction of exposed bedrock for steady hydraulic conditions, this distribution needs to be convolved with the probability distribution for bedrock for steady hydraulic conditions, this distribution

\[
\lambda_0 \text{ for in the investigated parameter range for large } \sigma, \text{ but it shows a maximum and subsequent decline for large } \mu. \text{ These trends of } \bar{m}_t \text{ with changing rate constants are as expected from equation (19), and in most cases equation (19) provides a good first-order estimate for the mean value of } m \text{ at steady state.}
\]

2.2. Bed Cover

Equations for bed cover have previously been proposed as deterministic models for steady state conditions when the bed configuration has already adjusted to supply conditions [Sklar and Dietrich, 1998; Turowski et al., 2007]. Turowski et al. [2007] used a probabilistic argument to derive an exponential equation for bed cover. Here I shall use a similar line of reasoning but model the stochastic variation explicitly. As such, the model proposed below provides a stochastic interpretation for the exponential model derived by Turowski et al. [2007].

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\]

2.3. Tools Effect, Particle Velocity, and Volume Removed Upon Each Impact

The model can be taken a step further by linking it to the bedrock erosion rate. In analogy with equation (1), the erosion rate \( E \) can be written as

\[
E = CnR_0.
\]

Here \( C \) is a constant incorporating the dependence on substrate properties and hydraulic conditions, for example, the velocity of the bed load particles and the volume of rock eroded at each impact. The linear dependence on the number of moving particle \( n \) models the tools effect [cf. Sklar and Dietrich, 2004], and other variables and constants in \( V_t \) and \( I_t \) have been subsumed into \( C \).

The two relevant variables probably exhibiting strongest fluctuations apart from the number of particles and their distribution on the bed are the travel velocity of the particles and the volume of rock removed upon a particle impact. Although there have been several attempts to develop theoretical models of rolling and saltating particles [Ancey et al., 2002, 2003; Lee et al., 2002; Niño and García, 1994; Wiberg and Smith, 1985], due to the strong nonlinearities in the fluid forces acting on the grains, none of these provides a good match with experimental data. Empirical equations have been proposed and often take a linear or power law relationship with the mean fluid velocity [Ancey et al., 2002; Fernandez Luque and van Beek, 1976; Niño et al., 1994; Seminara et al., 2002; Sklar and Dietrich, 2004], but these equations concentrate on the mean velocity and a stochastic distribution has not yet been developed.

The volume of rock eroded upon the impact of a single particle can fluctuate strongly, due to varying impact energy and angle, the shape of the impactor, its rotational motion, or the local erodibility of the substrate [Bitter, 1963a, 1963b]. Furthermore, typically several impacts are required at the same site to initiate erosion at all. Thus, even when the impact occurs under the exact same conditions, the volume removed can be expected to vary across several orders of magnitude. However, data on single particle impacts are scarce and a distribution function for the amount of erosion is not yet known.

The distribution function, for example, of the travel velocity or the volume of rock removed can be expected to be highly skewed. Schmeeckle et al. [2007] measured the velocity of the fluid approaching a particle in a laboratory flume and observed strong fluctuations. They reported values for skewness, which were positive, negative, or close to zero. It is currently unclear how these local variations in flow velocity scale up to a cross section or, in the case considered here, to a control volume. Lacking direct constraints, I model the variations of travel speed and volume removed (as well as other relevant variables) with a normal distribution; that is, \( C \) was assumed to be normally distributed with a mean value \( C = 8 \times 10^{-12} \text{ m/s} \) and a standard deviation of 0.25. These parameter values are essentially arbitrary and were chosen to give a mean erosion rate of the correct order of magnitude. The choice of a normal distribution will in effect just broaden the final distribution of erosion rates and hardly affect its shape. Since the number...
of particles within the control volume is a discrete parameter, only certain values for the erosion rate can be obtained, and often the same erosion rate is obtained for different states of the system with different probabilities of occurrence. Distributions for erosion rate have been obtained by classifying the erosion values in logarithmically spaced bins and summing the probabilities associated with them (Figure 7). The distributions show a wide spread of likely erosion rates over approximately an order of magnitude in both cases and are skewed toward higher erosion rates. For case S, the model predicts a higher mean erosion rate and a narrower distribution. The larger spread and lower mean erosion rate of case L is due to extended cover; on average ~20% of the bed is exposed in case L, in comparison with almost 60% in case S (compare Figure 6c).

3. Discussion

3.1. Stochastic Model of Bed Load Transport

[21] The model proposed herein is a generalization of the stochastic model of sediment transport developed by Ancey et al. [2008]. Since I assume that the number of stationary particles on the bed is not infinite but finite, the model reduces to Ancey et al.’s [2008] model for large m (Appendix A). A drawback of these stochastic models is the difficulty in constraining the rate constants. To directly measure them, detailed observations of the control volume are necessary, to the point of counting the arrival, evacuation, entrainment, and deposition of all particles within a time window. This is difficult to do in a controlled laboratory setting, and because of the high-energy environment in streams during bed load transport it seems hardly possibly to achieve in natural streams, especially at high discharges or transport rates. A further problem is that the rate constants are functions of the local hydraulic conditions [Ancey et al., 2008], and to make the use of such models practicable may require a large number of studies to cover all possible conditions. However, if a sufficiently large number of bed load measurements are available, the rate constants can be calibrated by fitting distribution functions. This does not provide a direct test of the model, but it may
serve to theoretical underpin the large variation in transport rates that is observed at constant hydraulic conditions [e.g., Ancey et al., 2006; Böhm et al., 2004; Gomez and Church, 1989; Kuhnle and Southard, 1988; Lisle et al., 2001; Métivier et al., 2004].

An important simplification of the stochastic model is the assumption of a single grain size and grain shape, resulting in constant rate parameters. Furthermore, the spatial heterogeneity of beds with mixed grain sizes can introduce another stochastic aspect into the sediment transport rates [Chen and Stone, 2008]. Similarly, grain arrangement and protrusion can vary at the same size distribution [Kirchner et al., 1990]. To take the different entrainment and deposition probabilities for grain mixtures into account, one would need to model the rate parameters as distribution functions, or keep track of the numbers of particles in each grain size class. This should be possible in principle, but needs further research in how different particle sizes interact in mixtures and how to model this interaction [cf. Hodge et al., 2007; Wilcock and Crowe, 2003]. This is beyond the scope of this work.

### 3.2. Stochastic Model of the Cover Effect

One of the most important simplifications in the stochastic cover model is probably that a single cell would not be able to hold all \( k \) particles, as is assumed in the model. Instead, if the number of particles within a cell reaches a threshold value, the sediment pile would collapse and the material would be redistributed into surrounding cells. Moreover, this threshold would be a function of the number of particles within neighboring cells; for example, a cell can hold more sediment particles if neighboring cells hold particles as well. A full treatment of the problem would need to use 3-D modeling of the dynamics of granular materials, a topic that is still incompletely understood. However, the probabilities predicted for a concentration of all particles in a few cells is small enough for this simplification to be negligible.

Previous models for the extent of cover were deterministic functions of supply and hydraulics, where a given set of boundary conditions leads to a unique state of bed cover. Currently the most frequently used model is the linear decline model [Sklar and Dietrich, 2004; Whipple and Tucker, 2002], where the fraction of exposed bedrock is assumed to decline linearly with increasing sediment supply until falling to zero when supply is equal to transport capacity (equation (2)). Alternatively, Turowski et al. [2007] derived an exponentially declining model (equation (3)). These models and the one proposed herein can be directly compared through the relationship between the time-averaged fraction of bed exposed \( R_a \) and the total number of particles in the control volume. This relationship is exponential with decay constant \( N \) (Figure 7), i.e.,

\[
\langle R_a \rangle = \exp \left\{ -\frac{k}{N} \right\}.
\]

Thus the model proposed herein provides a stochastic interpretation of the exponential formulation. Turowski et al. [2007] introduced the cover factor \( \varphi \) to account for regions of the bed that are intrinsically more or less favorable for deposition, e.g., because material collects in a local depression. If \( \varphi > 1 \), newly introduced sediment is more likely to fall on yet uncovered ground; if \( \varphi < 1 \), it is more likely to fall on already covered ground. If any position on the bed is an equally likely position for a particle, \( \varphi = 1 \). Including \( \varphi \), equation (23) becomes

\[
\langle R_a \rangle = \exp \left\{ -\varphi \frac{k}{N} \right\}.
\]

Equation (24) can be used to estimate \( \varphi \) for the experimental conditions described by Sklar and Dietrich [2001]. Assuming 2-D close packing of spheres on the bed at full cover, \( N \) can be estimated by the ratio of the total experimental area...
Using for the packing density of close packed spheres in a plane. (Figure 1), \( 8 \) would increase to be a larger one. A thicker layer is needed for full cover, the measured value of sufficient to completely protect the bed from erosion. If a calculation above assumes that a single layer of particles is would be expected for a flat bed under ideal conditions. The underlying processes. For example, experiment 2-A3 shows a standard deviation of 18% of the mean at steady state (taken to have occurred after 500 min), even though sediment supply is fairly steady. In general, the exponential model provides a good description of the experimental data (Figure 9). The cover factor \( \varphi \) is in all cases greater than one. This means that uncovered regions of the bed are more likely to be covered when additional sediment is introduced or, equivalently, that the extent of cover is greater than expected from a purely random distribution of sediment.

According to the model of Turowski et al. [2007], if \( Q_s < Q_t \), a reduction of the bed impacts can only occur if \( \varphi > 1 \). The observations at the Pitzbach by Turowski and Rickenmann [2009] thus imply that \( \varphi > 1 \) in this setting. The experimental results of Chatanantavet and Parker [2008] point into the same direction as Turowski and Rickenmann’s [2009] observations at the Pitzbach. This may be a real-world reflection of the model drawback mentioned above: At partial cover it seems virtually impossible that two or more grains lie on top of each other.

[25] Chatanantavet and Parker [2008] investigated how the fraction of exposed bedrock varies with discharge, sediment supply, and channel bed slope in a laboratory flume. They evaluated the linear decline model [Sklar and Dietrich, 1998] and concluded it to be appropriate to model the cover variation. The exponential model of Turowski et al. [2007] was not investigated. I will now compare the observation reported by Chatanantavet and Parker [2008] with the model developed above and discuss the implications.

[26] The extent of bed cover fluctuated strongly in the experiments, even at steady state [see Chatanantavet and Parker, 2008, Figures 8 and 9], confirming that a stochastic approach is necessary, and is able in principle to capture the underlying processes. For example, experiment 2-A3 shows

\[
M_N = \frac{N V_p \rho_s}{e} = \frac{\pi}{2\sqrt{3}} \frac{A_E}{eA_p} \rho_s = \frac{\pi}{3\sqrt{3}} \frac{A_E \rho_s d}{e}. \tag{25}
\]

Here \( d \) is the grain diameter and the factor \( \pi/2\sqrt{3} \) accounts for the packing density of close packed spheres in a plane. Using \( A_E = 120\pi \) cm\(^2\) and \( d = 6 \) mm results in \( M_N = 0.134 \) kg. Together with a decay constant equal to \((133.7 \pm 4.2) \times 10^{-3} \) kg obtained from a nonlinear fit to the experimental erosion curves reported by Sklar and Dietrich [2001] (Figure 1), \( \varphi = 1.01 \pm 0.03 \), which is close to \( \varphi = 1 \), which would be expected for a flat bed under ideal conditions. The calculation above assumes that a single layer of particles is sufficient to completely protect the bed from erosion. If a thicker layer is needed for full cover, the measured value of \( \varphi \) would increase to be a larger one.

\[ A_E \] and the projected area of a single particle \( A_p \). As the experimental data are given as masses, we need the mass of \( N/e \) particles \( M_N \), by multiplying \( N/e \) by the volume \( V_p \) of a particle and its density \( \rho_s \). The division by the base of the natural logarithm \( e \) is necessary since we are interested in the mass of the particles required to decrease the fraction of the exposed bed to \( 1/e \):

\[
\text{Figure 8. Mean fraction of exposed bed as function of } k = m + n. \text{ The relationship can be described by an exponential decay (equation (23)).}
\]
of particles within a control volume is small, bedrock is exposed and erosion can occur. The newly developed cover model predicts that the extent of bedrock coverage increases exponentially with the number of particles in the control volume, thus providing a stochastic interpretation for the cover model proposed by Turowski et al. [2007].

[30] Few data exist to directly evaluate the model assumption, and thus the discussions of the model results were focused on the cover effect. The fluctuations observed in the extent of bed cover at steady state in experiments [Chatanantavet and Parker, 2008] confirm that a stochastic approach is suitable and necessary. The exponential cover model provides a good description of experimental data. In general, the cover factor is larger than one. However, with the current data available it is not possible to finally discriminate between the exponential formulation and the linear cover model proposed previously [Sklar and Dietrich, 1998]. Since cover may vary stochastically even at constant water and sediment supply, it is necessary to observe the bed for some time at constant conditions to effectively constrain the statistics and the probability distributions. This may pose a problem in natural environments, as discharge of streams can vary suddenly and quickly at times when erosion occurs. More measurements both in the laboratory and in nature are necessary to test the proposed and other models. Observations of bed cover may also provide an alternative way to direct observations of entrainment and deposition to constrain the rate parameters in stochastic models of bed load transport.

Appendix A: Limit of Large m

[31] Here I demonstrate that the model derived herein is equivalent to Ancey et al.’s [2008] model in the limit of large m. Then, it is true that

$$m - 1 \approx m \approx m + 1.$$  

(A1)

Thus one can ignore the variation of bed particles $m$ and concentrate on the number of mobile particles $n$. Equation (11) can be rewritten as

$$\frac{\partial P(n,t)}{\partial t} = \lambda_0 P(n-1;t) + (m\lambda_1 + m(n-1)\mu)P(n-1;t) + (n+1)\sigma P(n+1;t) + (n+1)\nu P(n+1;t) - (\lambda_0 + m\lambda_1 + n\nu + n\sigma + mn\mu)P(n;t).$$  

(A2)

Since $m$ is approximately constant, $\lambda_1$ and $\mu$ can be redefined as

$$m\lambda_1 \rightarrow \lambda_1$$  

(A3)

$$m\mu \rightarrow \mu$$  

(A4)

Equation (A2) becomes

$$\frac{\partial P(n,t)}{\partial t} = \lambda_0 P(n-1;t) + (\lambda_1 + (n-1)\mu)P(n-1;t) + (n+1)\sigma P(n+1;t) + (n+1)\nu P(n+1;t) - (\lambda_0 + \lambda_1 + n\nu + n\sigma + n\mu)P(n;t).$$  

(A5)
Equation (A5) is the transition equation derived by Ancey et al. [2008, equation 2.8] assuming an infinite number of particles on the bed.

### Notation

- $A_e$: experimental bed area, m$^2$.
- $C$: constant dependent describing the relative erodibility, m$^{-1}$.
- $d$: representative grain size, m.
- $E$: erosion rate, m s$^{-1}$.
- $e$: base of the natural logarithm.
- $I_p$: particle impact rate per unit area, m$^{-2}$ s$^{-1}$.
- $k$: number of particles in control volume $k = m + n$.
- $M_n$: mass of $N/e$ particles, kg.
- $m$: number of stationary particles in control volume.
- $m_0$: expected value of $m$.
- $n$: number of moving particles in control volume.
- $n_0$: expected value of $n$.
- $P_k^x$: probability to observe $x$ empty cells at given $N$ and $k$.
- $P(m, n; t)$: probability to observe $m$ stationary and $n$ moving particles at time $t$.
- $Q_s$: bed load transport rate, m$^3$ s$^{-1}$.
- $Q_i$: bed load transport capacity, m$^3$ s$^{-1}$.
- $R_e$: fraction of exposed bed area.
- $t$: time, s.
- $V$: control volume, m$^3$.
- $V_p$: particle volume, m$^3$.
- $V_i$: average volume removed at each impact, s$^{-1}$.
- $x$: number of empty cells.
- $\rho_s$: density of sediment, kg m$^{-3}$.
- $\lambda_0$: upstream supply rate, s$^{-1}$.
- $\lambda_1$: entrainment rate by the flow, s$^{-1}$.
- $\mu$: entrainment rate by particle impact, s$^{-1}$.
- $\sigma$: deposition rate, s$^{-1}$.
- $\nu$: evacuation rate, s$^{-1}$.
- $\phi$: cover factor.

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### References


